

573. ON GENERALIZATION OF THE INEQUALITY OF A. OSTROWSKI AND SOME RELATED APPLICATIONS*

Gradimir V. Milovanović and Josip E. Pečarić

A. OSTROWSKI ([1]) has proved the following result:

Theorem A. Let f be a differentiable function on (a, b) and let, on (a, b) , $|f'(x)| \leq M$. Then, for every $x \in (a, b)$,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) (b-a) M.$$

G. V. MILOVANOVIĆ ([2]) has generalized the Theorem A when f is function of several variables. In the same paper [2], by use of Theorem A, the following theorem, which generalize the result of G. W. MACKEY ([3], [4, p. 297]), is proved:

Theorem B. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be a differentiable function defined on $[0, 1]$ and such that $|f'(x)| \leq M$ for $0 < x < 1$. Then

$$\left| \int_0^1 f(x) dx - \sum_{k=1}^n \lambda_k f(x_k) \right| \leq \frac{M}{2} \sum_{k=1}^n ((x_k - a_{k-1})^2 + (a_k - x_k)^2),$$

where

$$0 = a_0 < a_1 < \dots < a_n = 1$$

and

$$\lambda_k = a_k - a_{k-1}, \quad a_{k-1} \leq x_k \leq a_k \quad (k = 1, \dots, n).$$

In this paper we shall give generalizations of these results when $|f^{(n)}(x)| \leq M$ ($\forall x \in (a, b)$), and $n > 1$.

Theorem 1. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be n (> 1) times differentiable function such that $|f^{(n)}(x)| \leq M$ ($\forall x \in (a, b)$). Then, for every $x \in [a, b]$

$$\left| \frac{1}{n} \left(f(x) + \sum_{k=1}^{n-1} F_k \right) - \frac{1}{b-a} \int_a^b f(y) dy \right| \leq \frac{M}{n(n+1)!} \cdot \frac{(x-a)^{n+1} + (b-x)^{n+1}}{b-a},$$

where F_k is defined by

$$(1) \quad F_k \equiv F_k(f; n; x, a, b) \equiv \frac{n-k}{k!} \cdot \frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a}.$$

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Proof. Integrating by use of TAYLOR's formula

$$f(x) = f(y) + \sum_{k=1}^{n-1} \frac{1}{k!} f^{(k)}(y) (x-y)^k + \frac{1}{n!} f^{(n)}(\xi) (x-y)^n,$$

where $\xi = y + \theta(x-y)$ ($0 < \theta < 1$), we obtain

$$(2) \quad f(x)(b-a) - \int_a^b f(y) dy - \sum_{k=1}^{n-1} \frac{1}{k!} \int_a^b f^{(k)}(y) (x-y)^k dy = \frac{1}{n!} \int_a^b f^{(n)}(\xi) (x-y)^n dy.$$

Putting $I_k = \frac{1}{k!} \int_a^b f^{(k)}(y) (x-y)^k dy$ ($k \geq 1$) and $I_0 = \int_a^b f(y) dy$, by means of partial integration on I_k , we have

$$I_k = \frac{1}{k!} \{f^{(k-1)}(b)(x-b)^k - f^{(k-1)}(a)(x-a)^k\} + I_{k-1} \quad (1 \leq k \leq n-1),$$

i.e.

$$(n-k)(I_k - I_{k-1}) = -(b-a)F_k \quad (1 \leq k \leq n-1),$$

where F_k is defined by (1).

Since

$$\sum_{k=1}^{n-1} (n-k)(I_k - I_{k-1}) = -(b-a) \sum_{k=1}^{n-1} F_k,$$

i.e.

$$I_0 + \sum_{k=1}^{n-1} I_k = nI_0 - (b-a) \sum_{k=1}^{n-1} F_k,$$

it follows, from (2),

$$\begin{aligned} \left| f(x)(b-a) + (b-a) \sum_{k=1}^{n-1} F_k - nI_0 \right| &= \frac{1}{n!} \left| \int_a^b f^{(n)}(\xi) (x-y)^n dy \right| \leq \frac{M}{n!} \int_a^b |x-y|^n dy \\ &= \frac{M}{(n+1)!} (x-a)^{n+1} + (b-x)^{n+1}. \end{aligned}$$

Thus the proof is finished.

For $n=2$, the Theorem 1 reduces to the following corollary:

Corollary 1. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be twice differentiable function such that $|f''(x)| \leq M$ ($\forall x \in (a, b)$). Then

$$\left| \frac{1}{2} \left(f(x) + \frac{(x-a)f(a) + (b-x)f(b)}{b-a} \right) - \frac{1}{b-a} \int_a^b f(y) dy \right| \leq \frac{M(b-a)^2}{4} \left(\frac{1}{12} + \frac{\left(x - \frac{a+b}{2} \right)^2}{(b-a)^2} \right)$$

for every $x \in [a, b]$.

Now we shall give an application of Theorem 1. Namely, in the quadrature formula

$$(3) \quad \int_0^1 f(x) dx = \frac{1}{n} \sum_{j=1}^m \lambda_j f(x_j) + \frac{1}{n} \sum_{k=1}^{n-1} \sum_{j=1}^m F_k^{(j)} + R_m(f),$$

where

$$\begin{aligned} 0 &= a_0 < a_1 < \dots < a_m = 1, \\ \lambda_j &= a_j - a_{j-1}, \quad a_{j-1} \leq x_j \leq a_j \quad (j = 1, \dots, m), \\ F_k^{(j)} &\equiv \lambda_j F_j(f; n; x_j, a_{j-1}, a_j) \\ &\equiv \frac{n-k}{k!} \{(x_j - a_{j-1})^k f^{(k-1)}(a_{j-1}) - (x_j - a_j)^k f^{(k-1)}(a_j)\} \end{aligned}$$

we shall estimate $R_m(f)$.

Theorem 2. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be n (> 1) times differentiable function such that $|f^{(n)}(x)| \leq M$ ($\forall x \in (0, 1)$). For the error $R_m(f)$ in the quadrature formula (3), holds

$$(4) \quad |R_m(f)| \leq \frac{M}{n(n+1)!} \sum_{j=1}^m \{(x_j - a_{j-1})^{n+1} + (a_j - x_j)^{n+1}\}.$$

Proof. Let

$$E_j = \frac{1}{n} \left(f(x_j) + \sum_{k=1}^{n-1} \frac{1}{\lambda_j} F_k^{(j)} \right) - \frac{1}{\lambda_j} \int_{a_{j-1}}^{a_j} f(x) dx.$$

Since

$$|R_m(f)| = \left| \int_0^1 f(x) dx - \frac{1}{n} \sum_{j=1}^m \lambda_j f(x_j) - \frac{1}{n} \sum_{k=1}^{n-1} \sum_{j=1}^m F_k^{(j)} \right| = \left| \sum_{j=1}^m \lambda_j E_j \right| \leq \sum_{j=1}^m \lambda_j |E_j|,$$

on the basis of Theorem 1, we have

$$|R_m(f)| \leq \frac{M}{n(n+1)!} \sum_{j=1}^m \{(x_j - a_{j-1})^{n+1} + (a_j - x_j)^{n+1}\}.$$

Thus, the Theorem 2 is proved.

We now present some corollaries of the Theorem 2.

Corollary 2. If $x_j = a_j$ ($j = 1, \dots, m$) then

$$\begin{aligned} |R_m(f)| &= \left| \int_0^1 f(x) dx - \frac{1}{n} \sum_{j=1}^m \lambda_j f(a_j) - \frac{1}{n} \sum_{k=1}^{n-1} \frac{n-k}{k!} \sum_{j=1}^m \lambda_j^k f^{(k-1)}(a_{j-1}) \right| \\ &\leq \frac{M}{n(n+1)!} \sum_{j=1}^m \lambda_j^{n+1}. \end{aligned}$$

If, moreover, $a_j = \frac{j}{m}$ ($j = 1, \dots, m$) then

$$(5) \quad |R_m(f)| \leq \frac{M}{n(n+1)! m^n}.$$

Note that for $n=2$, (5) reduces to

$$|R_m(f)| = \left| \int_0^1 f(x) dx - \frac{1}{2m} \sum_{j=1}^m \left(f\left(\frac{j-1}{m}\right) + f\left(\frac{j}{m}\right) \right) \right| \leq \frac{M}{12m^2},$$

what represents the error of generalized trapezoidal rule (see [5], p. 95).

Corollary 3. If $x_j = \frac{1}{2}(a_{j-1} + a_j)$, (4) is reduced to

$$|R_m(f)| \leq \frac{M}{2^n n(n+1)!} \sum_{j=1}^m \lambda_j^{n+1}.$$

If, moreover, $a_j = \frac{j}{m}$ ($j = 1, \dots, m$) then

$$|R_m(f)| \leq \frac{M}{2^n m^n (n+1)!}.$$

In this case, formula (3) is reduced to

$$(6) \quad \int_0^1 f(x) dx = \frac{1}{nm} \sum_{j=1}^m f\left(\frac{2j-1}{2m}\right) + \frac{1}{n} \sum_{k=1}^{n-1} \frac{n-k}{(2m)^k k!} \sum_{j=1}^m \left(f^{(k-1)}\left(\frac{j-1}{m}\right) + (-1)^{k-1} f^{(k-1)}\left(\frac{j}{m}\right) \right) + R_m(f).$$

In special case for $n=3$, formula (6) becomes

$$\int_0^1 f(x) dx = \frac{1}{3m} \sum_{j=1}^m \left(f\left(\frac{j-1}{m}\right) + f\left(\frac{2j-1}{2m}\right) + f\left(\frac{j}{m}\right) \right) - \frac{1}{24m^2} (f'(1) - f'(0)) + R_m(f),$$

where

$$|R_m(f)| \leq \frac{1}{576 m^3} \max_{x \in (0,1)} |f'''(x)|.$$

REFERENCES

1. A. OSTROWSKI: *Über die Absolutabweichung einer differentierbaren Funktion von ihrem Integralmittelwert*. Comment. Math. Helv. **10** (1938), 226—227.
2. G. V. MILOVANOVIĆ: *On some integral inequalities*. These Publications № 498 — № 541 (1975), 119—124.
3. G. W. MACKEY: *The William Lowell Putnam Mathematical Competition*. Amer. Math. Monthly **54** (1947), 403.
4. D. S. MITRINOVIĆ (In cooperation with P. M. VASIĆ): *Analytic inequalities*. Berlin — Heidelberg — New York 1970.
5. F. B. HILDEBRAND: *Introduction to Numerical Analysis*. New York — Toronto — London 1974.