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An Exploration of the Versatile Nature of Enriched Contraction Mappings: Collage-Type Theorems and Inverse Problem Applications

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Abstract. This article delve into the theoretical and practical implications of enriched contraction mappings, including collage type theorems, inverse problem-solving methodologies, and the exploration of roughly enriched contractive mappings as perturbations of traditional enriched contractions. Finally, we highlight diverse real-world applications, demonstrating the practical efficacy derived from our theoretical elucidations.

1. Introduction

In the context of the present exposition, a nonempty set $\mathbb{X}$ and a self-mapping $\mathbb{T}$ from $\mathbb{X}$ to $\mathbb{X}$ are considered. The set of fixed points of $\mathbb{T}$, denoted as $\operatorname{Fix}(\mathbb{T})$, is defined as $\{x_* \in \mathbb{X} : \mathbb{T}x_* = x_*\}$. Furthermore, the nth iterate of $\mathbb{T}$ is defined in the conventional manner, with $\mathbb{T}^0$ representing the identity map ($\mathbb{I}$) and $\mathbb{T}^n$ expressed as $\mathbb{T}^{n-1} \circ \mathbb{T}$ for $n \geq 1$. The mapping $\mathbb{T}$ is referred to as a Picard operator, as explained in Rus [1], under two conditions: (i) $\operatorname{Fix}(\mathbb{T}) = \{p\}$ and (ii) the nth iteration of $\mathbb{T}$, denoted as $\mathbb{T}^n x_0$, converges to p as n approaches infinity for any initial point x_0 in $\mathbb{X}$. Of particular interest in the realm of nonlinear analysis are Picard operators falling under the broader category known as Picard–Banach contractions. Rooted in the contributions of Stefan Banach [3], the Banach Contraction Principle has long stood as a foundational theorem in the realm of mathematical analysis, playing a pivotal role in the examination of the convergence of iterated mappings within metric spaces. This principle stipulates that any contraction $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ in a complete metric space $(\mathbb{X}, d)$ – in other words, any mapping that satisfies the inequality $d(\mathbb{T}x, \mathbb{T}y) \leq \delta\, d(x, y)$ for all $x, y \in \mathbb{X}$, where δ lies within the interval $[0, 1)$ – is termed a Picard operator. The essence of this principle lies in establishing a robust mathematical framework ensuring the existence and uniqueness of solutions to diverse mathematical problems, particularly within fixed point contexts. The Banach contraction mapping principle, and its implications, has garnered significant attention for its theoretical and practical relevance, prompting efforts to expand its scope. The emergence of the "Collage

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Theorem" signifies a notable advancement resulting from these endeavors. Banach's Fixed Point Theorem serves as the foundation of the Collage Theorem, relying on the concept of "collage distance" $d(x, \mathbb{T}x)$ to quantify convergence towards solutions. Collage Theorems have proven effective in resolving inverse problems in various fields such as image compression, ecological modeling, and parameter estimation in differential equations, aiming to approximate desired elements within metric spaces through fixed point equations. Inverse problems in applied mathematics often involve approximating a designated element x within a complete metric space $(\mathbb{X}, d)$ via the fixed point x_* of a contraction mapping $\mathbb{T} : \mathbb{X} \to \mathbb{X}$. These problems are often approached through the lens of the "Collage Theorem," seeking a contraction mapping minimizing the collage distance, which serves as a regularization technique for addressing inverse problems.

Consider the following inverse problem: *Given an appropriate metric space* $(\mathbb{X}, d_{\mathbb{X}})$ *with* $x_{target} \in \mathbb{X}$ *and* $\overline{x}_\lambda \in \mathbb{X}$ *for all* $\lambda \in \Lambda$*, and* $\epsilon > 0$*, determine parameters* $\lambda \in \Lambda$ *such that the true approximation error* $d_{\mathbb{X}}\left(x_{target}, \overline{x}_\lambda\right) < \epsilon$.

The subsequent corollary of Banach's Theorem facilitates a practical reformulation of this inverse problem:

Theorem 1.1 (The Collage Theorem). *Suppose* $(\mathbb{X}, d_{\mathbb{X}})$ *is a complete metric space,* $\mathbb{T}$ *is a contraction mapping with fixed point* $\overline{x} \in \mathbb{X}$ *and contractivity factor* $\delta \in [0, 1)$*. Then for any target element* $x_{target} \in \mathbb{X}$*, we have*

$$d_{\mathbb{X}}\left(x_{target}, \overline{x}_\lambda\right) \leq \frac{1}{1-\delta} d_{\mathbb{X}}\left(x_{target}, \mathbb{T}x_{target}\right).$$

Leveraging the Collage Theorem with a set of contraction mappings $\mathbb{T}_\lambda$, each with fixed point $\overline{x}_\lambda$ and contractivity factor $\delta_\lambda \in [0, 1)$, where $\delta_\lambda < \delta < 1$, allows control of the true approximation error $d_{\mathbb{X}}\left(x_{\text{target}}, \overline{x}_\lambda\right)$ by minimizing the typically simpler calculation of the collage distance $d_{\mathbb{X}}\left(x_{\text{target}}, \mathbb{T}_\lambda x_{\text{target}}\right)$.

Driven by a desire for broader applicability and aiming to develop a more versatile toolkit to address challenges beyond the traditional scope, researchers have endeavored to enrich and generalize the class of contraction mappings by relaxing stringent contraction conditions. In the recent past, as a result of such efforts, Berinde and Păcurar [5] introduced a class of mappings known as "enriched contractions," which serves as a generalization of the Banach contraction mapping.

Definition 1.1. Let $(\mathbb{X}, \|\cdot\|)$ be a normed linear space. A mapping $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ is said to be a $(\theta, \gamma)-$enriched contraction if there exist $\theta \in [0, +\infty)$ and $\gamma \in [0, \theta + 1)$ such that

$$(\forall x, y \in \mathbb{X}) \quad \left\|\theta(x-y) + \mathbb{T}x - \mathbb{T}y\right\| \leq \gamma \left\|x-y\right\|. \tag{1}$$

Remark 1.1. (i) Every $(\theta, \gamma)-$enriched contraction $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ on a normed linear space is globally Lipschitz continuous, and hence continuous. In fact, inequality (1) implies that for all $x, y \in \mathbb{X}$,

$$\left\|\mathbb{T}x - \mathbb{T}y\right\| \leq \left\|\theta(x-y) + \mathbb{T}x - \mathbb{T}y\right\| + \theta\left\|x-y\right\| \leq (\theta+\gamma)\left\|x-y\right\|,$$

so $\mathbb{T}$ is $(\theta + \gamma)-$Lipschitz. While $\theta + \gamma$ may not be the minimal Lipschitz constant, it provides a uniform bound that is directly inherited from the enriched contraction property.

(ii) Berinde and Păcurar [5, Example 1] established that every classical contraction mapping $\mathbb{T}$ with a contractivity constant $\delta \in [0, 1)$ is a $(0, \delta)-$enriched contraction. We can strengthen this observation by showing that such mappings admit a broader parameterization. Indeed, if $\mathbb{T}$ is a $\delta-$contraction, then for any $\theta \geq 0$,

$$(\forall x, y \in \mathbb{X}) \quad \left\|\theta(x-y) + \mathbb{T}x - \mathbb{T}y\right\| \leq \theta\left\|x-y\right\| + \left\|\mathbb{T}x - \mathbb{T}y\right\| \leq (\theta+\delta)\left\|x-y\right\|.$$

Thus $\mathbb{T}$ satisfies the enriched contraction condition with any parameter pair (θ, γ) such that $\gamma \in [\theta + \delta, \theta + 1)$. The classical case $(\theta, \gamma) = (0, \delta)$ is therefore just one instance of a much richer family. In this sense, classical contractions naturally embed into the enriched contraction framework with a continuum of admissible parameter choices. This broader viewpoint not only recovers the earlier result but also provides additional flexibility in parameter selection without enlarging the underlying class of mappings, thereby offering a more comprehensive structural relationship between the two classes.

The example below illustrates a mapping that satisfies the criteria for an enriched contraction but not for a contraction mapping. Thus, it becomes clear that the class of enriched contractions indeed encompasses the class of contractions.

Example 1.1. Let us consider the Banach space $c_0 = \{\alpha = (\alpha_n)_n \subset \mathbb{K} : \lim_{n\to\infty} \alpha_n = 0\}$, equipped with the supremum norm, denoted as $\|\alpha\|_\infty = \sup_n |\alpha_n|$. Now, let us define a mapping $\mathbb{T} : c_0 \to c_0$ as follows:

$$\mathbb{T}\left((\alpha_n)_n\right) = \left(a_n\alpha_n - a_{n+1}\left|\alpha_{n+1}\right|\right)_n ,$$

where $a_n = -2^{-n}$ for all $n \in \mathbb{N}$. It can be easily observed that $\mathbb{T}$ is well-defined. On the other hand, considering that $|a_n| \le 1$ and $|a_{n+1}| \le 1/2$ for all $n \in \mathbb{N}$, it follows that for all $\alpha, \beta \in c_0$:

$$\left\|\mathbb{T}\alpha - \mathbb{T}\beta\right\|_\infty \le \frac{3}{2}\left\|\alpha - \beta\right\|_\infty .$$

In other words, the mapping $\mathbb{T}$ is a $(3/2)-$Lipschitzian mapping. However, for the sequences $\alpha = (1, 0, 0, \ldots)$ and $\beta = (0, 0, 1, 0, 0, \ldots)$ in c_0, it holds that $\left\|\mathbb{T}\alpha - \mathbb{T}\beta\right\|_\infty = \left\|\alpha - \beta\right\|_\infty = 1$. Hence, there is no $\delta \in [0, 1)$ such that $\left\|\mathbb{T}\alpha - \mathbb{T}\beta\right\|_\infty \le \delta\left\|\alpha - \beta\right\|_\infty$ holds for all $\alpha, \beta \in c_0$, indicating that the mapping $\mathbb{T}$ is not a contraction mapping.

We assert that $\mathbb{T}$ is a $(\theta, \theta + 1/2)-$enriched contraction mapping with $\theta > 1$. Utilizing the properties of absolute value and supremum, we derive the following inequality for all $\alpha = (\alpha_n)_n$, $\beta = (\beta_n)_n \in c_0$:

$$\left\|\theta(\alpha - \beta) + \mathbb{T}\alpha - \mathbb{T}\beta\right\|_\infty \le \sup_{n\in\mathbb{N}}\left|\theta - \frac{1}{2^n}\right|\left|\alpha_n - \beta_n\right| + \sup_{n\in\mathbb{N}}\left|-\frac{1}{2^{n+1}}\right|\left|\alpha_{n+1} - \beta_{n+1}\right|,$$

which further implies

$$\left\|\theta(\alpha - \beta) + \mathbb{T}\alpha - \mathbb{T}\beta\right\|_\infty \le \theta\sup_{n\in\mathbb{N}}\left|\alpha_n - \beta_n\right| + \frac{1}{2}\sup_{n\in\mathbb{N}}\left|\alpha_{n+1} - \beta_{n+1}\right|,$$

because, for $\theta > 1$, $\sup_{n\in\mathbb{N}}\left|\theta - \frac{1}{2^n}\right| < \theta$ and $\sup_{n\in\mathbb{N}}\left|-\frac{1}{2^{n+1}}\right| \le 1/2$. Additionally, as the sequence $(\alpha_{n+1} - \beta_{n+1})_n$ is a subsequence of $(\alpha_n - \beta_n)_n$, we have $\sup_{n\in\mathbb{N}}\left|\alpha_{n+1} - \beta_{n+1}\right| \le \sup_{n\in\mathbb{N}}\left|\alpha_n - \beta_n\right|$, and thus the above inequality simplifies to

$$\left\|\theta(\alpha - \beta) + \mathbb{T}\alpha - \mathbb{T}\beta\right\|_\infty \le \gamma\left\|\alpha - \beta\right\|_\infty ,$$

where $\gamma := \theta + 1/2$. Consequently, $\mathbb{T}$ is a $(\theta, \theta + 1/2)-$enriched contraction with $\theta > 1$.

Let us explore the following fixed-point problem:

Find an $x \in \mathbb{X}$ such that $\mathbb{T}x = x$, (2)

where $\mathbb{X}$ is a Banach space and $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ is a $(\theta, \gamma)-$enriched contraction.

In this article, our objectives encompass the following endeavors. In the subsequent section, we intend to demonstrate the well-posed nature of the problem delineated by (2). Following this, the third section will elucidate collage type theorems concerning $(\theta, \gamma)-$enriched contraction transformations, and will outline inverse problems pertaining to this class of mappings. Moving forward to the fourth chapter, we will introduce the class of $\rho-$roughly $(\theta, \gamma)-$enriched contractive mappings, which represents a perturbation of $(\theta, \gamma)-$enriched contraction transformations, and undertake an examination of the properties characterizing their invariant points. Lastly, the fifth chapter will encompass a diverse array of applications, including those addressing real-life challenges, with the aim of showcasing the practical efficacy derived from our theoretical elucidations.

2. Well Posedness and Ulam-Hyers Stability of Fixed Point Problem (2)

If a problem satisfies three fundamental criteria – existence, uniqueness, and stability – it is considered to be well-posed. This concept, initially analyzed by Tikhonov [7], has become a crucial guiding principle in mathematical analysis, particularly in the domain of optimization and variational problems, as demonstrated in Tikhonov's well-posedness example. Assessing the well-posedness of any mathematical or scientific problem is an essential step in understanding its structure and behavior. By verifying the existence of solutions, ensuring their uniqueness, and confirming their stability under perturbations, researchers can evaluate the robustness of a problem and the reliability of its solutions. Addressing these aspects not only strengthens the theoretical foundations of mathematical models but also supports the development of effective algorithms and methods for solving complex problems in various scientific disciplines.

The mathematical expression of well-posedness and stability may vary slightly depending on the context. However, in the analysis of the fixed point problem (2) in Banach spaces, the concepts of well-posedness and Ulam-Hyers stability—originally formulated in metric spaces by De Blasi and Myjak [8] and Rus [1] – are adapted for Banach spaces as under. These adaptations provide powerful and versatile tools for analyzing the well-posedness and stability of the fixed-point problem, offering a comprehensive framework for addressing its theoretical and practical aspects.

Definition 2.1 (see [8]). Consider a Banach space $(\mathbb{X}, \|\cdot\|)$ and a mapping $\mathbb{T} : \mathbb{X} \to \mathbb{X}$. The fixed point problem of $\mathbb{T}$ is considered well-posed if $\mathbb{T}$ has a unique fixed point $x_* \in \mathbb{X}$ and for any sequence $(x_n)_n$ in $\mathbb{X}$ with $\|x_n - \mathbb{T}x_n\| \to 0$ as $n \to \infty$, we have $\|x_n - x_*\| \to 0$ as $n \to \infty$.

Definition 2.2 (see [2]). Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space, and let $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ be a mapping. The fixed point problem (2) associated with (θ, γ)−enriched contraction mappings is said to exhibit Ulam-Hyers stable if there exists a constant $\kappa_{\mathbb{T}} > 0$ such that, for every $\varepsilon > 0$ and for any $w^* \in \mathbb{X}$ satisfying the inequality $\|w - \mathbb{T}(w)\| \leq \varepsilon$, there exists a solution x^* of the equation in (2) such that $\|w^* - x^*\| \leq \kappa_{\mathbb{T}}\varepsilon$.

Theorem 2.1. *The fixed point problem* (2) *for* (θ, γ) *−enriched contraction mappings in a Banach space* $\mathbb{X}$ *is well-posed.*

Proof. Let x_* be the unique fixed point of (θ, γ)−enriched contraction mapping $\mathbb{T}$, i.e., $\mathbb{T}x_* = x_*$. Let $(x_n)_n$ be a sequence in $\mathbb{X}$ such that $\|x_n - \mathbb{T}x_n\| \to 0$ as $n \to \infty$. Then, we have

$$\begin{aligned} \|x_* - x_n\| &\leq \left\| \frac{\theta}{\theta+1}(x_* - x_n) + \frac{1}{\theta+1}(\mathbb{T}x_* - \mathbb{T}x_n) \right\| + \left\| \frac{\theta}{\theta+1}x_n + \frac{1}{\theta+1}\mathbb{T}x_n - x_n \right\| \\ &= \frac{1}{\theta+1}\|\theta(x_* - x_n) + \mathbb{T}x_* - \mathbb{T}x_n\| + \frac{1}{\theta+1}\|\mathbb{T}x_n - x_n\| \\ &\leq \frac{1}{\theta+1}\gamma\|x_* - x_n\| + \frac{1}{\theta+1}\|\mathbb{T}x_n - x_n\| \quad \text{for all } n \geq 0. \end{aligned}$$

This implies that

$$\|x_* - x_n\| \leq \frac{1}{\theta+1-\gamma}\|\mathbb{T}x_n - x_n\| \quad \text{for all } n \geq 0,$$

which yields $\|x_* - x_n\| \to 0$ as $n \to \infty$, since $\|x_n - \mathbb{T}x_n\| \to 0$ as $n \to \infty$. □

Theorem 2.2. *The fixed point problem* (2) *for* (θ, γ) *−enriched contraction mappings in a Banach space* $\mathbb{X}$ *is Ulam-Hyers stable.*

Proof. Based on the result in [5, Theorem 2.4], it is established that $\mathrm{Fix}(\mathbb{T}) = \{x^*\}$. Let $\varepsilon > 0$, and consider $w^* \in \mathbb{X}$ such that $\|w^* - \mathbb{T}(w^*)\| \leq \varepsilon$. Using the triangle inequality for norms, we have

$$\|x^* - w^*\| \leq \frac{1}{\theta+1}\|\theta(x^* - w^*) + \mathbb{T}x_* - \mathbb{T}w^*\| + \frac{1}{\theta+1}\|\mathbb{T}w^* - w^*\|.$$

By applying the (θ,γ) −enriched contraction property of $\mathbb{T}$, it follows that

$$\|x^* - w^*\| \le \frac{\gamma}{\theta+1}\|w^* - x^*\| + \frac{\varepsilon}{\theta+1}.$$

Simplifying this inequality yields

$$\|x^* - w^*\| \le \kappa_{\mathbb{T}}\varepsilon,$$

where $\kappa_{\mathbb{T}} = 1/(\theta+1-\gamma)$. Hence, the fixed point problem (2) is Ulam-Hyers stable. □

3. Collage Type Theorems for Enriched Contractions

The classical Collage Theorem, formulated for (strict) contractions on complete metric spaces, underpins much of modern fractal analysis and fixed-point computation. By converting the search for a fixed point into the minimization of a collage distance, it yields both conceptual clarity and algorithmic leverage (e.g., in fractal image coding and parameter identification for iterated function systems). Nevertheless, the strict contractivity requirement is often too rigid for operators encountered in practice – particularly those whose "effective shrinkage" depends on state or context. This section develops collage-type bounds for enriched contractions, a family strictly larger than classical contractions. In doing so, we preserve the core correspondence between collage errors and fixed-point errors while relaxing the hypotheses to accommodate mappings with nonuniform or state-dependent contraction features. The resulting framework broadens the reach of collage methods in applications ranging from inverse problems and data-driven modeling to iterative approximation schemes in nonstandard settings.

Theorem 3.1. *Let $(\mathbb{X},\|\cdot\|)$ be a Banach space, and $\mathbb{T}:\mathbb{X}\to\mathbb{X}$ a (θ,γ) −enriched contraction mapping with fixed point $x_*\in\mathbb{X}$. For any $x\in\mathbb{X}$, the following inequalities hold*

$$\frac{1}{\theta+1+\gamma}\|\mathbb{T}x - x\| \le \|x - x_*\| \le \frac{1}{\theta+1-\gamma}\|\mathbb{T}x - x\|. \tag{3}$$

Proof. The proof is evident for the case of $x = x_*$. If $x \ne x_*$ for every $x\in\mathbb{X}$, then in this case, since $\mathbb{T}x_* = x_*$ and $\mathbb{T}$ is a (θ,γ) −enriched contraction, we obtain the following:

$$\begin{aligned}
\|\mathbb{T}x - x\| &= \|\mathbb{T}x + \theta x - (\theta+1)x\| \\
&= (\theta+1)\left\|\frac{1}{\theta+1}\mathbb{T}x + \frac{\theta}{\theta+1}x - x\right\| \\
&= (\theta+1)\left\|\frac{1}{\theta+1}\mathbb{T}x + \frac{\theta}{\theta+1}x - x_* + x_* - x\right\| \\
&\le \|\theta(x - x_*) + \mathbb{T}x - \mathbb{T}x_*\| + (\theta+1)\|x_* - x\| \\
&\le (\theta+1+\gamma)\|x_* - x\|,
\end{aligned}$$

which gives

$$\frac{1}{\theta+1+\gamma}\|\mathbb{T}x - x\| \le \|x - x_*\|. \tag{4}$$

On the flip side, by employing analogous arguments to that used previously, we derive the following inequality

$$\begin{aligned}
\|x - x_*\| &\le \left\|x - \frac{1}{\theta+1}\mathbb{T}x - \frac{\theta}{\theta+1}x\right\| + \left\|\frac{1}{\theta+1}\mathbb{T}x + \frac{\theta}{\theta+1}x - \frac{1}{\theta+1}\mathbb{T}x_* - \frac{\theta}{\theta+1}x_*\right\| \\
&= \left\|\left(1 - \frac{\theta}{\theta+1}\right)x - \frac{1}{\theta+1}\mathbb{T}x\right\| + \left\|\frac{\theta}{\theta+1}(x - x_*) + \frac{1}{\theta+1}(\mathbb{T}x - \mathbb{T}x_*)\right\| \\
&\le \frac{1}{\theta+1}\|x - \mathbb{T}x\| + \frac{\gamma}{\theta+1}\|x - x_*\|,
\end{aligned}$$

and this implies

$$\|x - x_*\| \leq \frac{1}{\theta + 1 - \gamma} \|x - \mathbb{T}x\| . \tag{5}$$

The desired inequality is achieved by combining the inequalities in (4) and (5). □

Example 3.1. Consider the Banach space $c_0 = \{x = (x_n)_n \subset \mathbb{K} : \lim_{n\to\infty} |x_n| = 0\}$, equipped with the supremum norm $\|x\|_\infty = \sup_n |x_n|$. Define the mapping $\mathbb{T} : B_{c_0} \to B_{c_0}$ on the unit ball $B_{c_0} = \{x \in c_0 : \|x\|_\infty \leq 1\}$ as follows

$$\mathbb{T}\Big((x_n)_n\Big) = \left(-\frac{3}{4^{n+1}} x_n + \frac{1}{4^{n+1}} |x_{n+1}|\right)_n .$$

Note that the mapping $\mathbb{T} : B_{c_0} \to B_{c_0}$ is well-defined and non-linear. Now, let us consider $x, y \in B_{c_0}$ and apply arguments similar to those used in Example 1.1. Consequently, we conclude that $\mathbb{T}$ is a $(\theta, \gamma)-$enriched contraction mapping with a fixed point $x_* = (0, 0, 0, \ldots) \in B_{c_0}$, where $\gamma := \theta + 1/4$ and $\theta > 3/4$. However, it is crucial to underscore that $\mathbb{T}$ falls short of being a contraction. Suppose there exists $\delta \in [0, 1)$ such that $\left\|\mathbb{T}x - \mathbb{T}y\right\|_\infty \leq \delta \left\|x - y\right\|_\infty$ for all $x, y \in B_{c_0}$. Taking $x = (1, 0, 0, 0 \ldots) \in B_{c_0}$ and $y = (0, 1, 0, 0, \ldots) \in B_{c_0}$, we observe that

$$\left\|\mathbb{T}x - \mathbb{T}y\right\|_\infty = 1 \leq \delta \left\|x - y\right\|_\infty = \delta,$$

resulting a contradiction. Hence, $\mathbb{T}$ cannot be a contraction.

Now, since $\theta > 3/4$ and $\theta + 1 + \gamma = 2\theta + 5/4 > 11/4$, we obtain the following for $x = (x_n)_n \in B_{c_0}$

$$\|\mathbb{T}x - x\|_\infty \leq \sup_{n\in\mathbb{N}} \left\{\left|-\frac{3}{4^{n+1}} - 1\right| |x_n| + \frac{1}{4^{n+1}} |x_{n+1}|\right\} \leq 2 \|x\|_\infty \leq (\theta + 1 + \gamma) \|x\|_\infty ,$$

and this implies

$$\frac{1}{\theta + 1 + \gamma} \|\mathbb{T}x - x\|_\infty \leq \|x - x_*\|_\infty .$$

Furthermore, for $x = (x_n)_n$, $y = (y_n)_n \in B_{c_0}$, the inequality $\sup_{n\in\mathbb{N}} \left|x_n + y_n\right| \geq \sup_{n\in\mathbb{N}} |x_n| - \inf_{n\in\mathbb{N}} \left|y_n\right|$ holds. Therefore, the following is obtained

$$\begin{aligned} \|\mathbb{T}x - x\|_\infty &\geq \sup_{n\in\mathbb{N}} \left|\left(\frac{3}{4^{n+1}} + 1\right) x_n\right| - \inf_{n\in\mathbb{N}} \left|\frac{1}{4^{n+1}} x_{n+1}\right| \\ &\geq \|x\|_\infty > \frac{3}{4} \|x\|_\infty = (\theta + 1 - \gamma) \|x - x_*\|_\infty , \end{aligned}$$

which gives

$$\frac{1}{\theta + 1 - \gamma} \|\mathbb{T}x - x\|_\infty \geq \|x - x_*\|_\infty .$$

Therefore, the inequality in (3) is fulfilled in the context of this example.

Theorem 3.2. *Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space, $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ a $(\theta, \gamma)-$enriched contraction mapping with fixed point $x_* \in \mathbb{X}$, and $\widetilde{\mathbb{T}} : \mathbb{X} \to \mathbb{X}$ a mapping with fixed point $\widetilde{x_*} \in \mathbb{X}$. Assume there exists constant $\epsilon > 0$ such that*

$$\sup_{x\in\mathbb{X}} \left\|\mathbb{T}x - \widetilde{\mathbb{T}}x\right\| \leq \epsilon.$$

Then, the following inequality holds

$$\left\|x_* - \widetilde{x_*}\right\| \leq \frac{\epsilon}{\theta + 1 - \gamma}. \tag{6}$$

Proof. From the assumptions $\mathbb{T}x_* = x_*$, $\widetilde{\mathbb{T}}\widetilde{x_*} = \widetilde{x_*}$, and $\mathbb{T}$ being a (θ, γ) −enriched contraction, we can derive the following

$$\begin{aligned}
\left\|x_* - \widetilde{x_*}\right\| &\leq \left\|\frac{1}{\theta+1}\mathbb{T}x_* + \frac{\theta}{\theta+1}x_* - \frac{1}{\theta+1}\mathbb{T}\widetilde{x_*} - \frac{\theta}{\theta+1}\widetilde{x_*}\right\| + \left\|\frac{1}{\theta+1}\mathbb{T}\widetilde{x_*} + \frac{\theta}{\theta+1}\widetilde{x_*} - \frac{1}{\theta+1}\widetilde{\mathbb{T}}\widetilde{x_*} - \frac{\theta}{\theta+1}\widetilde{x_*}\right\| \\
&= \frac{1}{\theta+1}\left\|\theta\left(x_* - \widetilde{x_*}\right) + \mathbb{T}x_* - \mathbb{T}\widetilde{x_*}\right\| + \frac{1}{\theta+1}\left\|\mathbb{T}\widetilde{x_*} - \widetilde{\mathbb{T}}\widetilde{x_*}\right\| \\
&\leq \frac{\gamma}{\theta+1}\left\|x_* - \widetilde{x_*}\right\| + \frac{1}{\theta+1}\left\|\mathbb{T}\widetilde{x_*} - \widetilde{\mathbb{T}}\widetilde{x_*}\right\|,
\end{aligned}$$

which leads to

$$\left\|x_* - \widetilde{x_*}\right\| \leq \frac{1}{\theta+1-\gamma}\sup_{x\in\mathbb{X}}\left\|\mathbb{T}x - \widetilde{\mathbb{T}}x\right\|,$$

resulting in

$$\left\|x_* - \widetilde{x_*}\right\| \leq \frac{\epsilon}{\theta+1-\gamma}. \qquad \square$$

Example 3.2. Let c_0, $\|\cdot\|_\infty$, B_{c_0}, and $\mathbb{T}$ be as defined in Example 3.1. Consider the mapping $\widetilde{\mathbb{T}} : B_{c_0} \to B_{c_0}$ described below

$$\widetilde{\mathbb{T}}\left((x_n)_n\right) = \begin{cases} -\frac{1}{4} - \frac{3}{4}x_0, & n = 0, \\ -\frac{3}{4^{n+1}}x_n, & n \geq 1. \end{cases}$$

It can be easily calculated that the fixed point of $\widetilde{\mathbb{T}}$ is $\widetilde{x_*} = (-1/7, 0, 0, 0, \ldots)$ and hence, $\left\|x_* - \widetilde{x_*}\right\|_\infty = 1/7$. For all $x \in B_{c_0}$, we obtain the following

$$\mathbb{T}x - \widetilde{\mathbb{T}}x = \underbrace{\left(\frac{1}{4}|x_1|, \frac{1}{4^2}|x_2|, \frac{1}{4^3}|x_3|, \ldots\right)}_{\alpha_n} + \underbrace{\left(\frac{1}{4}, 0, 0, \ldots\right)}_{\beta_n}.$$

Since $\alpha_n, \beta_n \geq 0$ and $\beta_n \leq \alpha_n + \beta_n$ for all $n \in \mathbb{N}$,

$$\frac{1}{4} = \sup_{n\in\mathbb{N}}\beta_n \leq \sup_{n\in\mathbb{N}}(\alpha_n + \beta_n) = \left\|\mathbb{T}x - \widetilde{\mathbb{T}}x\right\|_\infty.$$

Thus, with $\gamma = \theta + 1/4$ and $\epsilon = 1/4$, we have

$$\frac{1}{7} = \left\|x_* - \widetilde{x_*}\right\|_\infty \leq \frac{\epsilon}{\theta+1-\gamma} = \frac{1}{3},$$

which shows that the inequality in (6) is satisfied.

Theorem 3.3. *Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space, $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ a (θ, γ) −enriched contraction mapping with fixed point $x_* \in \mathbb{X}$, and $\widetilde{\mathbb{T}} : \mathbb{X} \to \mathbb{X}$ a mapping with fixed point $\widetilde{x_*} \in \mathbb{X}$. Assume tha*

$$\sup_{x\in\mathbb{X}}\left\|\mathbb{T}x - \widetilde{\mathbb{T}}x\right\| \leq \epsilon,$$

in which $\epsilon > 0$ is a constant. Then, the following inequality holds:

$$\left\|\widetilde{x_*} - \mathbb{T}\widetilde{x_*}\right\| \leq \left(\frac{\theta+1+\gamma}{\theta+1-\gamma}\right)\epsilon.$$

Proof. Since $\mathbb{T}$ is a $(\theta,\gamma)-$enriched contraction and $\mathbb{T}x_* = x_*$,

$$\begin{aligned}
\frac{1}{\theta+1}\left\|\widetilde{x}_* - \mathbb{T}\widetilde{x}_*\right\| &= \left\|\widetilde{x}_* - \frac{1}{\theta+1}\mathbb{T}\widetilde{x}_* - \frac{\theta}{\theta+1}\widetilde{x}_*\right\| \\
&= \left\|\widetilde{x}_* - x_* + x_* - \frac{1}{\theta+1}\mathbb{T}\widetilde{x}_* - \frac{\theta}{\theta+1}\widetilde{x}_*\right\| \\
&\leq \left\|\widetilde{x}_* - x_*\right\| + \frac{1}{\theta+1}\left\|\theta\left(x_* - \widetilde{x}_*\right) + \mathbb{T}x_* - \mathbb{T}\widetilde{x}_*\right\| \\
&\leq \frac{\theta+1+\gamma}{\theta+1}\left\|x_* - \widetilde{x}_*\right\|,
\end{aligned}$$

which is equivalent to

$$\left\|\widetilde{x}_* - \mathbb{T}\widetilde{x}_*\right\| \leq (\theta+1+\gamma)\left\|x_* - \widetilde{x}_*\right\|. \tag{7}$$

Substituting (6) into (7) and then applying the assumption $\sup_{x\in\mathbb{X}}\left\|\mathbb{T}x - \widetilde{\mathbb{T}}x\right\| \leq \epsilon$, we obtain

$$\left\|\widetilde{x}_* - \mathbb{T}\widetilde{x}_*\right\| \leq \left(\frac{\theta+1+\gamma}{\theta+1-\gamma}\right)\epsilon. \qquad \square$$

Example 3.3. Let c_0, $\|\cdot\|_\infty$, B_{c_0}, $\mathbb{T}$, and $\widetilde{\mathbb{T}}$ be as defined in Example 3.1. For $x \in B_{c_0}$, we have

$$\left\|\mathbb{T}x - \widetilde{\mathbb{T}}x\right\|_\infty = \sup_{n\in\mathbb{N}}\left\{\frac{1}{4} + \frac{1}{4}|x_1|, \frac{1}{4^2}|x_2|, \frac{1}{4^3}|x_3|, \ldots\right\} \leq \frac{1}{4} + \frac{1}{4}\|x\|_\infty,$$

from which we deduce

$$\sup_{x\in B_{c_0}}\left\|\mathbb{T}x - \widetilde{\mathbb{T}}x\right\|_\infty \leq \frac{1}{2} = \epsilon.$$

As $\widetilde{x}_* = (-1/7, 0, 0, 0, \ldots)$ and $\mathbb{T}$ is a $(\theta,\gamma)-$enriched contraction with $\theta > 3/4$ and $\gamma := \theta + 1/4$, we have $\left\|\widetilde{x}_* - \mathbb{T}\widetilde{x}_*\right\|_\infty = 1/4$ and $(\theta+1+\gamma)/(\theta+1-\gamma) = (8\theta+5)/3$. Hence, we obtain

$$\left\|\widetilde{x}_* - \mathbb{T}\widetilde{x}_*\right\| = \frac{1}{4} < \frac{5}{6} + \frac{8\theta}{6} = \left(\frac{8\theta+5}{3}\right)\cdot\frac{1}{2} = \left(\frac{\theta+1+\gamma}{\theta+1-\gamma}\right)\epsilon.$$

The connection between Collage-type results and inverse problems provides a principled route for approximating solutions: rather than solving complicated fixed-point equations directly, one reformulates them as tractable optimization objectives. Classical collage-based techniques have driven substantial progress in fractal image coding and in the study of dynamical systems, yet their reliance on uniform contractivity restricts applicability. In numerous practical settings – for instance adaptive control schemes, stochastic dynamical models, and nonautonomous systems – the contraction behaviour can vary with context or state and is not well represented by a single, global contraction constant. Extending the Collage framework to accommodate such nonuniform contraction phenomena produces a more flexible toolkit for inverse problems arising in these areas. This generalization supports stable approximation in neural architectures with non-Lipschitz components, in chaotic models exhibiting only transient regions of stability, and in data-driven models subject to structural uncertainty. Crucially, we prove that the essential link between a map's collage discrepancy and its fixed-point approximation error persists under these weakened contraction hypotheses, thereby preserving a favorable trade-off between computational practicability and theoretical control. By moving fixed-point techniques beyond the confines of strict uniform contractivity, the developments below aim to broaden the scope and precision of inverse-problem solvers in complex, adaptive environments.

We are confronted with the following inverse problem:

Problem 3.1. *Given an $\varepsilon > 0$ and a "target" $\overline{x} \in \mathbb{X}$, the objective is to find an (θ, γ) −enriched contraction mapping $\mathbb{T}_\varepsilon$ that possesses a unique fixed point $x^*_{\mathbb{T}_\varepsilon} = \mathbb{T}_\varepsilon\left(x^*_{\mathbb{T}_\varepsilon}\right)$, satisfying the condition $\left\|\overline{x} - x^*_{\mathbb{T}_\varepsilon}\right\| \leq \varepsilon$.*

Historically, this formulation is motivated by early work in fractal image coding (see [9]). Finding an operator whose exact fixed point minimizes the approximation error is in general highly challenging: the computational complexity of identifying optimal fractal operators has been shown to be prohibitive (cf. [10], NP-hardness results). Consequently, practical fractal coding methods typically sidestep direct fixed-point optimization and instead minimize the Collage distance $\|x - \mathbb{T}x\|$, a surrogate objective easier to handle computationally; this approach is commonly called Collage coding (see, e.g., [11]).

The Collage Theorem 3.1 allows us to reframe Problem 3.1 into the following equivalent, more tractable search.

Problem 3.2. *Given a $\delta > 0$ and a designated "target" point $\overline{x} \in \mathbb{X}$, the goal is to find an (θ, γ) −enriched contraction mapping $\mathbb{T}_\delta$ such that $\left\|\overline{x} - \mathbb{T}_\delta\overline{x}\right\| \leq \delta$.*

Thus, instead of directly seeking an enriched contraction whose fixed point is within ε of the prescribed target, one searches for a (θ, γ) −enriched contraction that maps the target to a point close to itself; the Collage bound then converts this property into a quantitative guarantee on the distance from the target to the operator's fixed point.

Proposition 3.1. *If a solution to Problem 3.2 exists, then there is also a solution to Problem 3.1.*

Proof. Let $\varepsilon > 0$ and $\overline{x} \in \mathbb{X}$ be given. Define $\delta := (\theta + 1 - \gamma)\,\varepsilon$, and let $\mathbb{T}_\delta$ be an enriched contraction mapping with $\left\|\overline{x} - \mathbb{T}_\delta\overline{x}\right\| \leq \delta$. If $\mathbb{T}_\delta\left(x^*_{\mathbb{T}_\delta}\right) = x^*_{\mathbb{T}_\delta}$, then according to Theorem 3.1, we have

$$\left\|x^*_{\mathbb{T}_\delta} - \overline{x}\right\| \leq \frac{1}{\theta + 1 - \gamma}\left\|\overline{x} - \mathbb{T}_\delta\overline{x}\right\| \leq \frac{1}{\theta + 1 - \gamma}\delta = \varepsilon. \qquad \square$$

4. Roughly Enriched Contractions

Inspired by the research conducted by Phu and Truong [12], we now pose the following question: What happens if the right-hand side of the inequality in (1) is perturbed by a positive number ρ? In this scenario, the mapping $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ satisfies the following condition: there exist $\theta \in [0, +\infty)$, $\gamma \in [0, \theta + 1)$, and $\rho > 0$, such that

$$(\forall x, y \in \mathbb{X}) \quad \left\|\theta\,(x - y) + \mathbb{T}x - \mathbb{T}y\right\| \leq \gamma\left\|x - y\right\| + \rho. \tag{8}$$

which is termed a (θ, γ, ρ) −roughly enriched contraction mapping. These mappings can occur in very natural circumstances. For example, when dealing with a (θ, γ, ρ) − roughly enriched contraction mapping $\mathbb{T}$ where an exact determination isn't feasible, one may resort to an approximation, denoted as $\widetilde{\mathbb{T}}$. For $\rho' = 2\max_{x\in\mathbb{X}}\left\|\mathbb{T}x - \widetilde{\mathbb{T}}x\right\|$, we have

$$\rho'' = 2\max_{x\in\mathbb{X}}\left\|\mathbb{T}_\tau x - \widetilde{\mathbb{T}}_\tau x\right\|,$$

where $\rho'' = \tau\rho'$, $\mathbb{T}_\tau x = (1 - \tau)\,x + \tau\mathbb{T}x$, $\widetilde{\mathbb{T}}_\tau x = (1 - \tau)\,x + \tau\widetilde{\mathbb{T}}x$, and $\tau \in (0, 1)$. Thus, the inequality in (1) yields

$$\begin{aligned}
\left\|\widetilde{\mathbb{T}}_\tau x - \widetilde{\mathbb{T}}_\tau y\right\| &\leq \left\|\widetilde{\mathbb{T}}_\tau x - \mathbb{T}_\tau x\right\| + \left\|\mathbb{T}_\tau x - \mathbb{T}_\tau y\right\| + \left\|\mathbb{T}_\tau y - \widetilde{\mathbb{T}}_\tau y\right\| \\
&\leq \left\|\mathbb{T}_\tau x - \mathbb{T}_\tau y\right\| + \rho'' \\
&\leq \tau\gamma\left\|x - y\right\| + \rho''
\end{aligned}$$

which implies

$$(\forall x, y \in \mathbb{X}) \quad \left\|\theta\,(x - y) + \widetilde{\mathbb{T}}x - \widetilde{\mathbb{T}}y\right\| \leq \gamma\left\|x - y\right\| + \frac{\rho''}{\tau},$$

where $\theta = (1-\tau)/\tau$. Therefore, $\widetilde{\mathbb{T}}$ is a $(\theta, \gamma, \rho''/\tau)$ −roughly enriched contraction mapping.

The following remark highlights the inclusiveness of the roughly enriched contraction framework. On the one hand, it subsumes the classical notion of roughly contractions as a special case ($\theta = 0$). On the other hand, it broadens the admissible parameter range by embedding every (δ, ρ) −roughly contraction into the more general (θ, γ, ρ) −roughly enriched contraction mappings class. Consequently, this extension not only preserves the classical setting but also provides additional flexibility in applications, where the interplay between the enrichment parameter θ and the contractivity factor γ may yield sharper or more adaptable convergence results.

Remark 4.1. (i) A (θ, γ, ρ) −roughly enriched contraction mapping reduces to a (δ, ρ) −roughly contraction mapping whenever $\theta = 0$, with $\gamma = \delta \in [0, 1)$ and $\rho > 0$.

(ii) Any (δ, ρ) −roughly contraction mapping, where $\delta \in [0, 1)$ and $\rho > 0$, can be regarded as a particular case of a (θ, γ, ρ) −roughly enriched contraction mapping by taking $\theta \in [0, +\infty)$ and $\gamma = \theta + \delta \in [\theta, \theta + 1)$.

(iii) As is immediately evident from the definition, every (θ, γ) −enriched contraction mapping is a (θ, γ, ρ) −roughly enriched contraction mapping.

The following example demonstrates that a (θ, γ, ρ) −roughly enriched contraction mapping is not necessarily an enriched contraction mapping.

Example 4.1. Let the space $\mathbb{X} = \mathbb{R}$ be endowed with the norm-induced usual metric. Define the mapping $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ as a Dirichlet-type function

$$\mathbb{T}x = \begin{cases} -x, & x \in \mathbb{Q}, \\ -x + 1, & x \in \mathbb{X} \backslash \mathbb{Q}. \end{cases}$$

Equivalently, $\mathbb{T}x = -x + \mathbb{S}x$ for all $x \in \mathbb{R}$, where

$$\mathbb{S}x = \begin{cases} 0, & x \in \mathbb{Q}, \\ 1, & x \in \mathbb{X} \backslash \mathbb{Q}. \end{cases}$$

The term $-x$ is continuous, but $\mathbb{S}x$ is discontinuous at every point. Consequently, $\mathbb{T}$ is discontinuous everywhere: for convergent rational sequences (x_n), $\mathbb{T}(x_n) \approx -x_n + 0$, while for irrational sequences (y_n), $\mathbb{T}(y_n) \approx -y_n + 1$. Function values for these distinct sequence types converge to different limits. Thus, $\mathbb{T}$ is discontinuous (indeed, discontinuous at every point) and by Remark 1.1, it cannot be a (θ, γ) −enriched contraction mapping. Nevertheless, $\mathbb{T}$ is a $(\theta, |\theta - 1|, 1)$ −roughly enriched contraction mapping, as shown below.

Case 1: If $x, y \in \mathbb{Q}$ (or $x, y \in \mathbb{X} \backslash \mathbb{Q}$), then $\mathbb{T}x = -x$ and $\mathbb{T}y = -y$ (or $\mathbb{T}x = -x + 1$ and $\mathbb{T}y = -y + 1$). For any $x, y \in \mathbb{Q}$ (or $x, y \in \mathbb{X} \backslash \mathbb{Q}$) and arbitrary $\rho > 0$, we have

$$\left|\theta(x - y) + \mathbb{T}x - \mathbb{T}y\right| = |\theta - 1| \left|x - y\right| \leq |\theta - 1| \left|x - y\right| + \rho.$$

Selecting

$$\gamma = |\theta - 1| = \begin{cases} \theta - 1, & \theta \geq 1, \\ -\theta + 1, & \theta \in [0, 1), \end{cases}$$

ensures $\gamma \in [0, \theta + 1)$ for any $\theta \in [0, +\infty)$. Hence, for these cases, $\mathbb{T}$ is a $(\theta, |\theta - 1|, \rho)$−roughly enriched contraction mapping.

Case 2: If $x \in \mathbb{Q}$ and $y \in \mathbb{X} \backslash \mathbb{Q}$ (or $y \in \mathbb{Q}$ and $x \in \mathbb{X} \backslash \mathbb{Q}$), then $\mathbb{T}x = -x$ and $\mathbb{T}y = -y + 1$ (or $\mathbb{T}y = -y$ and $\mathbb{T}x = -x + 1$). For any $x \in \mathbb{Q}$ and $y \in \mathbb{X} \backslash \mathbb{Q}$ (or $y \in \mathbb{Q}$ and $x \in \mathbb{X} \backslash \mathbb{Q}$), we have

$$\left|\theta(x - y) + \mathbb{T}x - \mathbb{T}y\right| = \left|\theta(x - y) - x + y - 1\right| \leq |\theta - 1| \left|x - y\right| + 1.$$

Choosing $\gamma = |\theta - 1|$ and $\rho = 1$, it follows that for any $\theta \in [0, +\infty)$, $\mathbb{T}$ is a $(\theta, |\theta - 1|, 1)$ −roughly enriched contraction mapping in these cases.

$(\theta,\gamma,\rho)-$roughly enriched contraction mappings may not invariably have fixed points, even when the conditions for the set $\mathbb{X}$ are identical to those in [5, Theorem 2.4]. However, they do accommodate what are known as $\mu-$fixed, or $\mu-$invariant points, denoted as ω^*, which fulfill the condition

$$\|\omega^* - \mathbb{T}\omega^*\| \leq \mu$$

for some $\mu > 0$. The set of all $\mu-$invariant points of $\mathbb{T}$ will be denoted by $\Omega_{\mu,\mathbb{T}}$.

Remark 4.2. It is readily apparent that $\Omega_{\mu,\mathbb{T}} \subset \Omega_{\mu,\mathbb{T}_\tau}$. Indeed, since $\tau \in (0,1)$, for every $\omega^* \in \Omega_{\mu,\mathbb{T}}$, we have the following

$$\|\omega^* - \mathbb{T}_\tau\omega^*\| = \tau\,\|\omega^* - \mathbb{T}\omega^*\| \leq \tau\mu \leq \mu.$$

Example 4.2. Let the space $\mathbb{X} = \mathbb{R}$ be endowed with the usual metric induced by the norm. Define the mapping $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ as

$$\mathbb{T}x = \begin{cases} -x+\pi, & x \in \mathbb{Q}, \\ -x, & x \in \mathbb{X}\backslash\mathbb{Q}. \end{cases}$$

Using arguments similar to those in Example 4.1, it can be readily verified that:
(i) $\mathbb{T}$ is discontinuous (in fact, discontinuous at every point);
(ii) $\mathbb{T}$ is not a $(\theta,\gamma)-$enriched contraction mapping as also follows from Remark 1.1;
(iii) $\mathbb{T}$ is a $(\theta,|\theta-1|,\pi)-$roughly enriched contraction mapping.
On the other hand, solving the equation $\mathbb{T}x = x$ for $x \in \mathbb{Q}$ and $x \in \mathbb{X}\backslash\mathbb{Q}$ yields $x = \pi/2 \notin \mathbb{Q}$ and $x = 0 \notin \mathbb{X}\backslash\mathbb{Q}$, respectively. This demonstrates that $\mathbb{T}$ has no fixed point.

Theorem 4.1. *Let $(\mathbb{X},\|\cdot\|)$ be a Banach space, and $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ a $(\theta,\gamma,\rho)-$roughly enriched contraction mapping. Let $\{x_n\}$ be a sequence defined by Krasnosel'skiǐ iteration algorithm*

$$x_0 \in \mathbb{X}, \quad x_{n+1} = (1-\tau)\,x_n + \tau\mathbb{T}x_n, \quad n \in \mathbb{N}, \tag{9}$$

where $\tau = (\theta+1)^{-1}$. Suppose

$$\kappa := \|x_0 - \mathbb{T}x_0\| - \frac{\rho}{1-\tau\gamma} > 0.$$

(i) *If $\mu > \rho/(1-\tau\gamma)$ and $n \geq \log_{\tau\gamma}\left((\mu - \rho/(1-\tau\gamma))\,\kappa^{-1}\right)$, then x_n provided by* (9) *is a $\mu-$invariant point under both $\mathbb{T}$ and $\mathbb{T}_\tau$.*

(ii) *If c^* is a cluster point of the sequence $\{x_n\}$, then it is a $\mu-$invariant point under $\mathbb{T}_\tau$ and $\mathbb{T}$, respectively with $\mu = \tau\rho/(1-\tau\gamma)$ and $\mu = \rho/(1-\tau\gamma)$.*

(iii) *For every $\mu > 0$, the sets $\Omega_{\mu,\mathbb{T}}$ and $\Omega_{\mu,\mathbb{T}_\tau}$ are bounded. If $\mu \geq \tau\rho/(1-\tau\gamma)$, then $\mathbb{T}_\tau\Omega_{\mu,\mathbb{T}_\tau} \subset \Omega_{\mu,\mathbb{T}_\tau}$. If $\mu \geq \tau\rho/(1-\tau^2\gamma)$, then $\mathbb{T}_\tau\Omega_{\mu,\mathbb{T}} \subset \Omega_{\mu,\mathbb{T}_\tau}$.*

Proof. We will split the proof into two parts.

Case 1. $\theta > 0$. It is evident that $\tau \in (0,1)$ as $\tau = (\theta+1)^{-1}$ and the inequality from (8) transforms as follows

$$(\forall x,y \in \mathbb{X}) \qquad \left\|\left(\frac{1}{\tau}-1\right)(x-y) + \mathbb{T}x - \mathbb{T}y\right\| \leq \gamma\left\|x-y\right\| + \rho,$$

which can be rewritten equivalently as

$$(\forall x,y \in \mathbb{X}) \qquad \left\|\mathbb{T}_\tau x - \mathbb{T}_\tau y\right\| \leq \tau\gamma\left\|x-y\right\| + \tau\rho. \tag{10}$$

As $\gamma \in [0, \theta + 1)$, $\tau\gamma \in (0, 1)$, and therefore $\mathbb{T}_\tau$ becomes a $(\tau\gamma, \tau\rho)-$roughly contraction mapping. Given $\mathbb{T}_\tau x = (1-\tau)x + \tau\mathbb{T}x$, the Krasnosel'skiĭ iteration algorithm described in (9) corresponds to the Picard iteration algorithm associated with $\mathbb{T}_\tau$, namely,

$$x_0 \in \mathbb{X}, \quad x_{n+1} = \mathbb{T}_\tau x_n, \quad n \in \mathbb{N}. \tag{11}$$

(i) By using (10) and (11), we obtain the following:

$$\begin{aligned}
\|\mathbb{T}_\tau x_{n-1} - \mathbb{T}_\tau x_n\| &\leq \tau\gamma \|x_{n-1} - x_n\| + \tau\rho \\
&\leq (\tau\gamma)^2 \|x_{n-2} - x_{n-1}\| + (\tau\gamma + 1)\tau\rho \\
&\leq (\tau\gamma)^3 \|x_{n-3} - x_{n-2}\| + \left((\tau\gamma)^2 + \tau\gamma + 1\right)\tau\rho \\
&\vdots \\
&\leq (\tau\gamma)^n \|x_0 - x_1\| + \left((\tau\gamma)^{n-1} + \cdots + \tau\gamma + 1\right)\tau\rho \\
&= (\tau\gamma)^n \|x_0 - \mathbb{T}_\tau x_0\| + \frac{1-(\tau\gamma)^n}{1-\tau\gamma}\tau\rho,
\end{aligned}$$

which, given that $\mu > \rho/(1-\tau\gamma)$ and $n \geq n_0 := \log_{\tau\gamma}\left((\mu - \rho/(1-\tau\gamma))\,\kappa^{-1}\right)$, leads to

$$\begin{aligned}
\|x_n - \mathbb{T}x_n\| &\leq (\tau\gamma)^n \left[\|x_0 - \mathbb{T}x_0\| - \frac{\rho}{1-\tau\gamma}\right] + \frac{\rho}{1-\tau\gamma} \\
&= (\tau\gamma)^n \kappa + \frac{\rho}{1-\tau\gamma} \\
&\leq \mu - \frac{\rho}{1-\tau\gamma} + \frac{\rho}{1-\tau\gamma} \\
&= \mu,
\end{aligned} \tag{12}$$

thus, x_n is $\mu-$invariant under $\mathbb{T}$.

On the other hand, given that $\mu > \rho/(1-\tau\gamma)$, $n \geq n_0$, and $\tau \in (0,1)$, it follows that

$$\mu - \frac{\tau\rho}{1-\tau\gamma} > \mu - \frac{\rho}{1-\tau\gamma} \geq (\tau\gamma)^n \kappa \geq (\tau\gamma)^n \tau\kappa.$$

Therefore, we derive the following

$$\begin{aligned}
\|x_n - \mathbb{T}_\tau x_n\| &\leq (\tau\gamma)^n \|x_0 - \mathbb{T}_\tau x_0\| + \frac{1-(\tau\gamma)^n}{1-\tau\gamma}\tau\rho \\
&= (\tau\gamma)^n \tau \left[\|x_0 - \mathbb{T}x_0\| - \frac{\rho}{1-\tau\gamma}\right] + \frac{\tau\rho}{1-\tau\gamma} \\
&= (\tau\gamma)^n \tau\kappa + \frac{\tau\rho}{1-\tau\gamma} \\
&\leq \mu - \frac{\tau\rho}{1-\tau\gamma} + \frac{\tau\rho}{1-\tau\gamma} \\
&= \mu.
\end{aligned}$$

This implies that x_n remains $\mu-$invariant under the mapping $\mathbb{T}_\tau$.

(ii) For any arbitrary $n \geq 1$, we have the following:

$$\begin{aligned}
\|c^* - \mathbb{T}_\tau c^*\| &\leq \|c^* - x_n\| + \|\mathbb{T}_\tau x_{n-1} - \mathbb{T}_\tau c^*\| \\
&\leq \|c^* - x_n\| + \tau\gamma \|x_{n-1} - c^*\| + \tau\rho \\
&\leq \|c^* - x_n\| + \tau\gamma \left[\|x_{n-1} - \mathbb{T}_\tau x_{n-1}\| + \|\mathbb{T}_\tau x_{n-1} - c^*\|\right] + \tau\rho \\
&= (1+\tau\gamma)\|x_n - c^*\| + \tau\gamma \|x_{n-1} - \mathbb{T}_\tau x_{n-1}\| + \tau\rho,
\end{aligned}$$

which leads to

$$\|c^* - \mathbb{T}_\tau c^*\| \le (1 + \tau\gamma) \|x_n - c^*\| + \tau^2\gamma \|x_{n-1} - \mathbb{T}x_{n-1}\| + \tau\rho,$$

and

$$\|c^* - \mathbb{T}c^*\| \le \frac{1 + \tau\gamma}{\tau} \|x_n - c^*\| + \tau\gamma \|x_{n-1} - \mathbb{T}x_{n-1}\| + \rho.$$

From the inequality in (12), we obtain

$$\limsup_{n\to\infty} \|x_{n-1} - \mathbb{T}x_{n-1}\| \le \frac{\rho}{1 - \tau\gamma}.$$

Since c^* is a cluster point of the sequence $\{x_n\}$, considering a subsequence converging to c^* leads to

$$\begin{aligned}\|c^* - \mathbb{T}_\tau c^*\| &\le \tau^2\gamma\frac{\rho}{1 - \tau\gamma} + \tau\rho \\ &= \left(\frac{\tau\gamma}{1 - \tau\gamma} + 1\right)\tau\rho \\ &= \frac{\tau\rho}{1 - \tau\gamma},\end{aligned}$$

and

$$\begin{aligned}\|c^* - \mathbb{T}c^*\| &\le \tau\gamma\frac{\rho}{1 - \tau\gamma} + \rho \\ &= \frac{\rho}{1 - \tau\gamma},\end{aligned}$$

which means c^* is a $\mu-$invariant point under $\mathbb{T}_\tau$ and $\mathbb{T}$, respectively with $\mu = \tau\rho/(1-\tau\gamma)$ and $\mu = \rho/(1-\tau\gamma)$.

(iii) By Remark 4.2, we have the inclusion $\Omega_{\mu,\mathbb{T}} \subset \Omega_{\mu,\mathbb{T}_\tau}$. Consequently it suffices to prove boundedness of $\Omega_{\mu,\mathbb{T}_\tau}$; boundedness of $\Omega_{\mu,\mathbb{T}}$ then follows immediately.

Let $x, y \in \Omega_{\mu,\mathbb{T}_\tau}$. Applying (10) and the defining bounds $\|x - \mathbb{T}_\tau x\| \le \mu$, $\left\|\mathbb{T}_\tau y - y\right\| \le \mu$, we obtain

$$\begin{aligned}\left\|x - y\right\| &\le \|x - \mathbb{T}_\tau x\| + \left\|\mathbb{T}_\tau x - \mathbb{T}_\tau y\right\| + \left\|\mathbb{T}_\tau y - y\right\| \\ &\le \mu + \mu + \tau\gamma \left\|x - y\right\| + \tau\rho.\end{aligned}$$

Rearranging gives

$$\left\|x - y\right\| \le \frac{2\mu + \tau\rho}{1 - \tau\gamma},$$

thus confirming that $\Omega_{\mu,\mathbb{T}_\tau}$ is bounded. By Remark 4.2 we then have boundedness of $\Omega_{\mu,\mathbb{T}}$. Indeed, any two points $x, y \in \Omega_{\mu,\mathbb{T}}$ satisfy $\left\|x - y\right\| \le (2\tau\mu + \tau\rho)/(1 - \tau\gamma)$.

We now check the stated invariance inclusions. For any $x \in \Omega_{\mu,\mathbb{T}_\tau}$, we have

$$\left\|\mathbb{T}_\tau x - \mathbb{T}_\tau^2 x\right\| \le \tau\gamma \|x - \mathbb{T}_\tau x\| + \tau\rho \le \tau\gamma\mu + \tau\rho.$$

Given that $\mu \ge \tau\rho/(1 - \tau\gamma)$, we get

$$\left\|\mathbb{T}_\tau x - \mathbb{T}_\tau^2 x\right\| \le \tau\gamma\mu + \mu (1 - \tau\gamma) = \mu,$$

which implies $\mathbb{T}_\tau x \in \Omega_{\mu,\mathbb{T}_\tau}$. Therefore, for $\mu \ge \tau\rho/(1 - \tau\gamma)$, it follows that $\mathbb{T}_\tau\Omega_{\mu,\mathbb{T}_\tau} \subset \Omega_{\mu,\mathbb{T}_\tau}$.

Similarly, for any $x \in \Omega_{\mu,\mathbb{T}} \subset \Omega_{\mu,\mathbb{T}_\tau}$, we have

$$\left\|\mathbb{T}_\tau x - \mathbb{T}_\tau^2 x\right\| \le \tau\gamma \|x - \mathbb{T}_\tau x\| + \tau\rho \le \tau^2\gamma\mu + \tau\rho.$$

Given $\mu \geq \tau\rho/(1-\tau^2\gamma)$, we get

$$\left\|\mathbb{T}_\tau x - \mathbb{T}_\tau^2 x\right\| \leq \tau^2\gamma\mu + \left(1-\tau^2\gamma\right)\mu = \mu,$$

implying $\mathbb{T}_\tau x \in \Omega_{\mu,\mathbb{T}_\tau}$. Therefore, for $\mu \geq \tau\rho/(1-\tau^2\gamma)$, it holdsthat $\mathbb{T}_\tau\Omega_{\mu,\mathbb{T}} \subset \Omega_{\mu,\mathbb{T}_\tau}$.

Case 2. $\theta = 0$. In this case, we have $\tau = 1$, and therefore $\mathbb{T}_\tau = \mathbb{T}$. Consequently, the proposed iteration reduces to the standard Picard iteration. Moreover, when $\theta = 0$, the inequality (10) takes the form $\left\|\mathbb{T}x - \mathbb{T}y\right\| \leq \delta\left\|x-y\right\| + \rho$ for all $x, y \in \mathbb{X}$ which shows that $\mathbb{T}$ is a $(\delta,\rho)-$roughly contraction mapping. The fixed point results for this class of mappings have already been established in detail by [12, Theorem 2.1]. Hence, in our setting, the statements (i)–(iii) remain valid with $\tau = 1$ and the corresponding constants adjusted accordingly. □

Next example illustrates how a discontinuous mapping without fixed points can nevertheless be handled within the framework of roughly enriched contractions. Krasnosel'skiĭ iteration provides sequences converging to approximate fixed points, in the sense of $\mu-$invariance, with explicit geometric decay rates. This confirms the practical utility of the extended theory in treating highly irregular operators beyond the classical contraction setting.

Example 4.3. Let $\mathbb{X}$ and $\mathbb{T}:\mathbb{X}\to\mathbb{X}$ be defined as in Example 4.2, where $\mathbb{T}$ is a $(\theta, |\theta-1|, \pi)-$roughly enriched contraction mapping. Fix $\theta \geq 0$ and set

$$\tau = \frac{1}{1+\theta}, \quad \mathbb{T}_\tau x = (1-\tau)x + \tau\mathbb{T}x,$$

so that the Krasnosel'skiĭ iteration reads $x_{n+1} = \mathbb{T}_\tau x_n$. By the definition of the $\mathbb{T}$, one has $\rho = \pi$. For the specific case where we set $\theta = 2$, the associated parameters are determined to be

$$\tau = \frac{1}{3} \quad \text{and} \quad \gamma = |\theta - 1| = 1,$$

hence

$$\tau\gamma = \frac{1}{3}, \quad \frac{\rho}{1-\tau\gamma} = \frac{3\pi}{2} \approx 4.71238898038469, \quad \frac{\tau\rho}{1-\tau\gamma} = \frac{\pi}{2} \approx 1.5707963267948966.$$

A direct computation yields the explicit branchwise formula for $\mathbb{T}_\tau$

$$\mathbb{T}_\tau x = \begin{cases} (1-2\tau)x + \tau\pi, & x \in \mathbb{Q}, \\ (1-2\tau)x, & x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$

With $\tau = 1/3$ this becomes

$$\mathbb{T}_\tau x = \begin{cases} \frac{1}{3}(x+\pi), & x \in \mathbb{Q}, \\ \frac{1}{3}x, & x \in \mathbb{R}\backslash\mathbb{Q}, \end{cases}$$

and $|1-2\tau| = 1/3 < 1$, so each branch is affine-contraction.

Two concrete initial data will be used to verify the hypotheses and conclusions of Theorem 4.1: $x_0 = \sqrt{101} \in \mathbb{R}\backslash\mathbb{Q}$ and $x_0 = 10 \in \mathbb{Q}$. The residuals with respect to $\mathbb{T}$ and $\mathbb{T}_\tau$ are as follows, respectively

$$\text{res}_\mathbb{T}(x) := \|x - \mathbb{T}x\| = \begin{cases} |2x-\pi|, & x \in \mathbb{Q}, \\ 2|x|, & x \in \mathbb{R}\backslash\mathbb{Q}, \end{cases} \qquad \text{res}_{\mathbb{T}_\tau}(x) = \tau\,\text{res}_\mathbb{T}(x). \tag{13}$$

Thus,

- if $x_0 = \sqrt{101}$ then $\|x_0 - \mathbb{T}x_0\| = 2\sqrt{101} \approx 20.09975$, and with
$$\frac{\rho}{1-\tau\gamma} = \frac{3\pi}{2}, \quad \kappa = \|x_0 - \mathbb{T}x_0\| - \frac{\rho}{1-\tau\gamma} = 2\sqrt{101} - \frac{3\pi}{2} \approx 15.387 > 0;$$

- If $x_0 = 10$ then $\|x_0 - \mathbb{T}x_0\| = |20 - \pi| \approx 16.8584$, and with
$$\frac{\rho}{1-\tau\gamma} = \frac{3\pi}{2}, \quad \kappa = 20 - \frac{5\pi}{2} \approx 12.146 > 0.$$

Hence the required $\kappa > 0$ holds in both cases. To illustrate statement (i) of Theorem 4.1 choose, for example, $\mu = 5$ (note $\mu > 3\pi/2$). The theorem gives the threshold

$$n \geq n_0 = \log_{\tau\gamma}\left(\left(\mu - \frac{\rho}{1-\tau\gamma}\right)\kappa^{-1}\right).$$

For $x_0 = \sqrt{101}$ we compute $(\mu - \rho/(1-\tau\gamma))\,\kappa^{-1} \approx 0.01869138$ and hence $n \geq \log_{1/3}(0.01869138) \approx 3.622$, so $n = 4$ suffices; for $x_0 = 10$ the right-hand side is ≈ 3.407, again giving $n = 4$. The explicit upper bound provided by the proof,

$$\|x_n - \mathbb{T}x_n\| \leq (\tau\gamma)^n\,\kappa + \frac{\rho}{1-\tau\gamma},$$

evaluated at $n = 4$ gives $(1/3)^4\,\kappa + 3\pi/2 \approx 4.902 \leq 5$, while the actual residuals computed above are ≈ 0.2481 and ≈ 0.3245, respectively; therefore x_4 is a $\mu-$invariant point for both $\mathbb{T}$ and $\mathbb{T}_\tau$, verifying (i) concretely.

For a given iteration x_n we have the simple dichotomy

$$\mathbb{T}x_n = \begin{cases} -x_n + \pi, & x_n \in \mathbb{Q}, \\ -x_n, & x_n \in \mathbb{R}\backslash\mathbb{Q} \end{cases} \quad \Longrightarrow \quad x_{n+1} = \mathbb{T}_\tau x_n = \begin{cases} \frac{1}{3}(x_n + \pi), & x_n \in \mathbb{Q}, \\ \frac{1}{3}x_n, & x_n \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$

If an iteration is irrational, then it remains in the irrational branch under the map $x \mapsto x/3$; if an iteration is rational then the next iteration is $(x_n + \pi)/3$, which is irrational because $\pi \in \mathbb{R}\backslash\mathbb{Q}$. Thus, after at most one step the geometric law $x_{n+1} = x_n/3$ governs the subsequent dynamics and the sequence tends to 0. To further illustrate the behavior of the Krasnosel'skiĭ iteration applied to the discontinuous mapping $\mathbb{T}$ introduced above, we provide several explicit numerical simulations. In particular, we consider the two starts and different choices of the parameter θ (hence $\tau = (\theta+1)^{-1}$). The resulting sequences $\{x_n\}$ are accompanied by the corresponding residuals $\mathrm{res}_{\mathbb{T}}(x_n)$ and $\mathrm{res}_{\mathbb{T}_\tau}(x_n)$, which quantify the degree of $\mu-$invariance under $\mathbb{T}$ and the averaged operator $\mathbb{T}_\tau$. The four scenarios considered in this context exhibit distinct dynamical behaviors and can be naturally grouped into two conceptual classes.

Convergence Cases (A and B):

- **Case A** ($\theta = 2$, $\tau = 1/3$, $x_0 = \sqrt{101}$) demonstrates geometric decay of the residuals, with both $\mathrm{res}_{\mathbb{T}}$ and $\mathrm{res}_{\mathbb{T}_\tau}$ converging rapidly to zero from an irrational initial point.
- **Case B** ($\theta = 2$, $\tau = 1/3$, $x_0 = 10$) illustrates that a rational starting point quickly transitions into the irrational branch, after which geometric convergence similar to Case A ensues, albeit with a distinct first iterate reflecting the discontinuity of the mapping.

Numerical iteration results for Cases A and B, illustrating convergence behavior of the mappings $\mathbb{T}$ and $\mathbb{T}\tau$, are presented in Table 1. For $\theta = 2$ and $\tau = 1/3$, x_n together with the residuals $\mathrm{res}\mathbb{T}(x_n)$ and $\mathrm{res}\mathbb{T}\tau(x_n)$ for two distinct initial conditions: an irrational starting point $x_0 = \sqrt{101}$ (Case A) and a rational starting point $x_0 = 10$ (Case B). In both cases, the residuals exhibit geometric decay toward zero, confirming rapid convergence to an approximate fixed point. Case B additionally highlights the effect of the rational-irrational discontinuity at the first iteration.

Two-Cycle Cases (C and D):

Table 1: Numerical iteration results for Cases A and B illustrating convergence behavior of the mappings $\mathbb{T}$ and $\mathbb{T}\tau$. For $\theta = 2$ and $\tau = 1/3$. x_n together with the residuals $\mathrm{res}\mathbb{T}(x_n)$ and $\mathrm{res}\mathbb{T}\tau(x_n)$ for two distinct initial conditions: an irrational starting point $x_0 = \sqrt{101}$ (Case A) and a rational starting point $x_0 = 10$ (Case B). In both cases, the residuals exhibit geometric decay toward zero, confirming rapid convergence to an approximate fixed point. Case B additionally highlights the effect of the rational-irrational discontinuity at the first iteration.

	Case A: $\theta = 2,\ \tau = 1/3,\ x_0 = \sqrt{101}$			Case B: $\theta = 2,\ \tau = 1/3,\ x_0 = 10$		
n	x_n	$\mathrm{res}_{\mathbb{T}}\,(x_n)$	$\mathrm{res}_{\mathbb{T}_\tau}\,(x_n)$	x_n	$\mathrm{res}_{\mathbb{T}}\,(x_n)$	$\mathrm{res}_{\mathbb{T}_\tau}\,(x_n)$
0	10.0498756211	20.0997512422	6.6999170807	10.0000000000	16.8584073464	5.6194691155
1	3.3499585404	6.6999170807	2.2333056936	4.3805308829	8.7610617658	2.9203539219
2	1.1166528468	2.2333056936	0.7444352312	1.4601769609	2.9203539219	0.9734513073
3	0.3722176156	0.7444352312	0.2481450771	0.4867256536	0.9734513073	0.3244837691
4	0.1240725385	0.2481450771	0.0827150257	0.1622418845	0.3244837691	0.1081612564
5	0.0413575128	0.0827150257	0.0275716752	0.0540806282	0.1081612564	0.0360537521
6	0.0137858376	0.0275716752	0.0091905584	0.0180268761	0.0360537521	0.0120179174
7	0.0045952792	0.0091905584	0.0030635195	0.0060089587	0.0120179174	0.0040059725
8	0.0015317597	0.0030635195	0.0010211732	0.0020029862	0.0040059725	0.0013353242
9	0.0005105866	0.0010211732	0.0003403911	0.0006676621	0.0013353242	0.0004451081
10	0.0001701955	0.0003403911	0.0001134637	0.0002225540	0.0004451081	0.0001483694

- **Case C** ($\theta = 1$, $\tau = 1/2$, $x_0 = \sqrt{101}$) and **Case D** ($\theta = 1$, $\tau = 1/2$, $x_0 = 10$) exhibit qualitatively different behavior. The iterates settle into a persistent two-cycle oscillation between 0 and $\pi/2$, and the residuals stabilize at strictly positive values (π and $\pi/2$, respectively), thus remaining permanently bounded away from zero.

Taken together, these cases offer compelling evidence in support of Theorem 4.1, showing that approximate fixed points naturally emerge even when exact fixed points do not exist, while simultaneously demonstrating that the enriched contraction framework can generate either rapid convergence or sustained oscillatory behavior, contingent on the selected parameters and initial conditions.

Statement (ii) of Theorem 4.1 concerns cluster points: as just observed $x_n \to 0$ in both cases, so every cluster point is $c^* = 0$. For this point $\mathbb{T}0 = \pi$ (because $0 \in \mathbb{Q}$), hence $\|0 - \mathbb{T}0\| = \pi \approx 3.1415926536 \leq 3\pi/2$ and $\mathbb{T}_\tau 0 = \tau\pi = \pi/3$ so $\|0 - \mathbb{T}_\tau 0\| = \pi/3 \approx 1.0471975512 \leq \pi/2$; consequently 0 satisfies the μ−bounds claimed in (ii) (the theorem gives upper bounds and our concrete values lie strictly below those bounds).

Finally, (iii) asserts boundedness of the sets $\Omega_{\mu,\mathbb{T}} = \left\{x : \|x - \mathbb{T}x\| \leq \mu\right\}$ and $\Omega_{\mu,\mathbb{T}_\tau}$ and two invariance inclusions at the thresholds $\mu \geq \tau\rho/(1-\tau\gamma) = \pi/2$ and $\mu \geq \tau\rho/(1-\tau^2\gamma) = 3\pi/8 \approx 1.178097$. By the definition of $\mathrm{res}_{\mathbb{T}}(x)$, we have

$$\Omega_{\mu,\mathbb{T}} \subset \left[\frac{\pi-\mu}{2}, \frac{\pi+\mu}{2}\right] \cup \left[\frac{\pi}{2}, \frac{\pi}{2}\right] \quad \text{for every } \mu > 0,$$

which is a bounded (indeed compact) subset of $\mathbb{R}$. Moreover, since $\mathrm{res}_{\mathbb{T}_\tau}(x) = \tau \mathrm{res}_{\mathbb{T}}(x)$ we have $\Omega_{\mu,\mathbb{T}_\tau} = \Omega_{\mu/\tau,\mathbb{T}}$, which is bounded for every $\mu > 0$. The two inclusions stated in (iii) can be established by distinguishing between rational and irrational inputs.

1. Verification of the inclusion $\mathbb{T}_\tau\Omega_{\mu,\mathbb{T}_\tau} \subset \Omega_{\mu,\mathbb{T}_\tau}$ for $\mu \geq \pi/2$.

Take any $x \in \Omega_{\mu,\mathbb{T}_\tau}$, so that $\|x - \mathbb{T}_\tau x\| \leq \mu$. We consider separately the irrational and rational cases.

- Case $x \in \mathbb{R}\backslash\mathbb{Q}$: Here one has $\mathbb{T}x = -x$, and consequently $\mathbb{T}_\tau x = (1-2\tau)x$. A straightforward computation shows that

$$\left\|\mathbb{T}_\tau x - \mathbb{T}_\tau^2 x\right\| = |1-2\tau|\,\|x - \mathbb{T}_\tau x\| \leq |1-2\tau|\,\mu.$$

  Specializing to $\tau = 1/3$, the inequality becomes $\mu/3 \leq \mu$, which clearly holds for all $\mu > 0$.

- Case $x \in \mathbb{Q}$: In this setting we have $\mathbb{T}x = -x + \pi$, hence $\mathbb{T}_\tau x = (1-2\tau)\,x + \tau\pi$. By a similar estimate,

  $$\left\|\mathbb{T}_\tau x - \mathbb{T}_\tau^2 x\right\| = |1-2\tau|\,\|x - \mathbb{T}_\tau x\| \le |1-2\tau|\,\mu.$$

  Again, with $\tau = 1/3$, this reduces to $\mu/3 \le \mu$, which is trivially satisfied for arbitrary $\mu > 0$.

2. Verification of the inclusion $\mathbb{T}_\tau\Omega_{\mu,\mathbb{T}} \subset \Omega_{\mu,\mathbb{T}_\tau}$ for $\mu \ge 3\pi/8$.

Consider $x \in \Omega_{\mu,\mathbb{T}}$, i.e., $\|x - \mathbb{T}x\| \le \mu$. As before, the argument branches into irrational and rational inputs.

- Case $x \in \mathbb{R}\backslash\mathbb{Q}$: Here $\mathbb{T}x = -x$, which enforces $|x| \le \mu/2$, while $\mathbb{T}_\tau x = (1-2\tau)\,x$. It follows that

  $$\left\|\mathbb{T}_\tau x - \mathbb{T}_\tau^2 x\right\| = 2\tau\,|1-2\tau|\,|x| \le \tau\,|1-2\tau|\,\mu.$$

  Inserting $\tau = 1/3$ gives $\mu/9 \le \mu$, valid for all $\mu > 0$.

- Case $x \in \mathbb{Q}$: In this case, $\mathbb{T}x = -x + \pi$, hence $|2x - \pi| \le \mu$, and $\mathbb{T}_\tau x = (1-2\tau)\,x + \tau\pi$. Then

  $$\left\|\mathbb{T}_\tau x - \mathbb{T}_\tau^2 x\right\| = \tau\,|1-2\tau|\,|2x-\pi| \le \tau\,|1-2\tau|\,\mu.$$

  Once more, taking $\tau = 1/3$ yields $\mu/9 \le \mu$, which is automatically satisfied for any $\mu > 0$.

Thus, the inclusions $\mathbb{T}_\tau\Omega_{\mu,\mathbb{T}_\tau} \subset \Omega_{\mu,\mathbb{T}_\tau}$ and $\mathbb{T}_\tau\Omega_{\mu,\mathbb{T}} \subset \Omega_{\mu,\mathbb{T}_\tau}$ are verified. In particular, although conditions $\mu \ge \pi/2$ and $\mu \ge 3\pi/8$ are typically imposed to ensure non-emptiness of the approximate fixed point set, the above inequalities themselves remain valid for all $\mu > 0$.

In conclusion, with the concrete choices $\theta = 2$, $\tau = 1/3$, $\gamma = 1$, $\rho = \pi$ and either $x_0 = \sqrt{101}$ or $x_0 = 10$, every hypothesis of Theorem 4.1 is satisfied, the finite-index μ−invariance in (i) occurs for small n (e.g., $n = 4$ for $\mu = 5$), the unique cluster point 0 meets the bounds in (ii), and the boundedness and invariance claims of (iii) hold numerically and analytically for the thresholds given above.

Remark 4.3. The arguments above naturally lead to the following two questions: (i) Under what conditions does the inclusion $\mathbb{T}\Omega_{\mu,\mathbb{T}} \subseteq \Omega_{\mu,\mathbb{T}}$ hold? (ii) What is the nature of the relationship between the sets $\mathbb{T}_\tau\Omega_{\mu,\mathbb{T}_\tau}$ and $\mathbb{T}_\tau\Omega_{\mu,\mathbb{T}}$?

We now proceed to tackle these questions.

(i) Let $\omega^* \in \Omega_{\mu,\mathbb{T}}$, that is, $\|\omega^* - \mathbb{T}\omega^*\| \le \mu$. Substituting $x = \omega^*$ and $y = \mathbb{T}\omega^*$ into (8) yields

$$\left\|\theta\,(\omega^* - \mathbb{T}\omega^*) + \mathbb{T}\omega^* - \mathbb{T}^2\omega^*\right\| \le \gamma\,\|\omega^* - \mathbb{T}\omega^*\| + \rho.$$

By the triangle inequality, this implies

$$\left\|\mathbb{T}\omega^* - \mathbb{T}^2\omega^*\right\| \le (\theta + \gamma)\,\|\omega^* - \mathbb{T}\omega^*\| + \rho.$$

Since $\|\omega^* - \mathbb{T}\omega^*\| \le \mu$, we obtain the key bound

$$\left\|\mathbb{T}\omega^* - \mathbb{T}^2\omega^*\right\| \le (\theta + \gamma)\mu + \rho. \tag{14}$$

In order for $\mathbb{T}\omega^* \in \Omega_{\mu,\mathbb{T}}$ to hold for every $\omega^* \in \Omega_{\mu,\mathbb{T}}$ we require $\left\|\mathbb{T}\omega^* - \mathbb{T}^2\omega^*\right\| \le \mu$. By inequality (14), this condition is satisfied whenever

$$(\theta + \gamma)\mu + \rho \le \mu,$$

that is,

$$\mu \ge \frac{\rho}{1 - (\theta + \gamma)}, \quad \text{provided } 1 - (\theta + \gamma) > 0. \tag{15}$$

Hence, if $\theta + \gamma < 1$ and μ is chosen such that $\mu \ge \rho/\left(1 - (\theta + \gamma)\right)$, then indeed $\mathbb{T}\Omega_{\mu,\mathbb{T}} \subseteq \Omega_{\mu,\mathbb{T}}$.

We now comment on degenerate cases:

- If $1-(\theta+\gamma)\le 0$ (i.e., $\theta+\gamma\ge 1$), then no finite μ satisfies the inequality unless unless $\rho=0$.
- If $\rho=0$ and $\theta+\gamma\le 1$, inequality (14) simplifies to $\left\|\mathbb{T}\omega^*-\mathbb{T}^2\omega^*\right\|\le(\theta+\gamma)\mu\le\mu$, so invariance holds for all $\mu>0$, i.e., $\mathbb{T}\Omega_{\mu,\mathbb{T}}\subseteq\Omega_{\mu,\mathbb{T}}$ unconditionally.
- If $\rho>0$ but $\theta+\gamma\ge 1$, then condition $(\theta+\gamma)\,\mu+\rho\le\mu$ cannot be met by any finite μ, since left–hand side grows faster. In such cases, invariance cannot be guaranteed by this direct approach.

For instance, consider the space $\mathbb{X}$ and the mapping $\mathbb{T}$ from Example 4.2. From the definition of $\mathrm{res}_{\mathbb{T}}(x)$ given in (13), we obtain

$$\Omega_{\mu,\mathbb{T}}=\underbrace{\left\{\omega^*\in\mathbb{R}\setminus\mathbb{Q}:2\,|\omega^*|\le\mu\right\}}_{\text{irrational part }\left(-\mu/2,\mu/2\right)\cap\mathbb{R}\setminus\mathbb{Q}}\cup\underbrace{\left\{\omega^*\in\mathbb{Q}:|2\omega^*-\pi|\le\mu\right\}}_{\text{rational points near }\pi/2}.$$

Since $\mathbb{T}$ is a $(\theta,|\theta-1|\,,\pi)$ –roughly enriched contraction mapping, we have

$$\theta+\gamma=\theta+|\theta-1|=\begin{cases}1, & \theta\in[0,1],\\ 2\theta-1>1, & \theta\in(1,\infty).\end{cases}$$

Hence, for every $\theta\ge 0$, it follows that $\theta+\gamma\ge 1$. In this situation, the sufficient condition for invariance stated in (15) can never be fulfilled, because $1-(\theta+\gamma)\le 0$. Consequently, the straightforward global guarantee fails in this case.

In summary, the favorable regime is when $\theta+\gamma<1$. It is worth emphasizing that this restriction is stronger than the standard enriched contractivity requirement $\gamma<\theta+1$, as it demands the sum $\theta+\gamma$ to be strictly less than one.

(ii) It is readily observed that $\Omega_{\mu,\mathbb{T}_\tau}=\Omega_{\mu/\tau,\mathbb{T}}$ holds for $\tau\in(0,1)$. Since $\mu/\tau\ge\mu$ for $\tau\in(0,1)$, we have $\Omega_{\mu/\tau,\mathbb{T}}\supseteq\Omega_{\mu,\mathbb{T}}$. Applying $\mathbb{T}_\tau$ (a deterministic mapping) preserves this inclusion:

$$\mathbb{T}_\tau\left(\Omega_{\mu/\tau,\mathbb{T}}\right)\supseteq\mathbb{T}_\tau\left(\Omega_{\mu,\mathbb{T}}\right).$$

Substituting the identity $\Omega_{\mu,\mathbb{T}_\tau}=\Omega_{\mu/\tau,\mathbb{T}}$ yields

$$\mathbb{T}_\tau\Omega_{\mu,\mathbb{T}_\tau}\supseteq\mathbb{T}_\tau\Omega_{\mu,\mathbb{T}}.$$

We now show that this inclusion is in fact strict under the standing hypotheses (in particular for the concrete choice $\tau=1/3$) and when $\mu\ge\pi/2$.

To prove strictness, it suffices to produce an element of $\mathbb{T}_\tau\Omega_{\mu,\mathbb{T}_\tau}$ that does not belong to $\mathbb{T}_\tau\Omega_{\mu,\mathbb{T}}$. Work on the irrational branch, where $\mathbb{T}x=-x$ and hence $\|x-\mathbb{T}x\|=2\,|x|$. For an irrational x the membership relations read

$$x\in\Omega_{\mu,\mathbb{T}}\iff|x|\le\frac{\mu}{2},\quad x\in\Omega_{\mu/\tau,\mathbb{T}}\iff|x|\le\frac{\mu}{2\tau}.$$

For $\tau\in(0,1)$, the interval $(\mu/2,\mu/(2\tau)]$ has strictly positive length $(1-\tau)\mu/(2\tau)$, hence is nonvoid. Choose any irrational x^* satisfying

$$\frac{\mu}{2}<|x^*|\le\frac{\mu}{2\tau}\ \text{ and }\ x^*\notin\pi+\mathbb{Q}:=\left\{\pi+q:q\in\mathbb{Q}\right\}.$$

Such a choice is always possible because the irrationals are uncountable and dense in $\mathbb{R}$, while $\pi+\mathbb{Q}$ is a countable set. Avoiding this set ensures that the rational branch cannot produce a rational preimage that would conflict with our construction. Consequently, $x^*\in\Omega_{\mu/\tau,\mathbb{T}}=\Omega_{\mu,\mathbb{T}_\tau}$ but $x^*\notin\Omega_{\mu,\mathbb{T}}$. Define $y:=\mathbb{T}_\tau x^*$; since $x^*\in\Omega_{\mu,\mathbb{T}_\tau}$, we have $y\in\mathbb{T}_\tau\Omega_{\mu,\mathbb{T}_\tau}$. Assume, toward a contradiction, that $y\in\mathbb{T}_\tau\Omega_{\mu,\mathbb{T}}$. Then there exists $z\in\Omega_{\mu,\mathbb{T}}$ with $\mathbb{T}_\tau z=y$. Two cases arise:

1. If $z \in \mathbb{R}\setminus\mathbb{Q}$, then $\mathbb{T}_\tau z = (1-2\tau)\,z$, so $(1-2\tau)\,z = (1-2\tau)\,x^*$, whence $z = x$. This contradicts $x^* \notin \Omega_{\mu,\mathbb{T}}$.
2. If $z \in \mathbb{Q}$, then $\mathbb{T}_\tau z = (1-2\tau)\,z + \tau\pi$, so

$$(1-2\tau)\,z + \tau\pi = (1-2\tau)\,x^* \Longrightarrow z = x^* - \frac{\tau}{1-2\tau}\pi.$$

For $\tau = 1/3$, this simplifies to $z = x^* - \pi$. By construction, $x^* \notin \pi + \mathbb{Q}$, which implies $z = x^* - \pi \notin \mathbb{Q}$. This contradicts the assumption $z \in \mathbb{Q}$.

Both alternatives lead to contradictions; therefore no such z exists, and $y \notin \mathbb{T}_\tau\Omega_{\mu,\mathbb{T}}$. We have thus produced an explicit witness $y \in \mathbb{T}_\tau\Omega_{\mu,\mathbb{T}_\tau}\setminus\mathbb{T}_\tau\Omega_{\mu,\mathbb{T}}$, proving that

$$\mathbb{T}_\tau\Omega_{\mu,\mathbb{T}_\tau} \supsetneq \mathbb{T}_\tau\Omega_{\mu,\mathbb{T}}$$

whenever $\mu \geq \pi/2$ (and in particular for $\tau = 1/3$). This demonstrates that the image under $\mathbb{T}_\tau$ of the larger $\mathbb{T}$−sublevel set $\Omega_{\mu/\tau,\mathbb{T}}$ properly contains the image of the smaller one $\Omega_{\mu,\mathbb{T}}$; equivalently, the inclusion is strict and the two images do not coincide.

5. Applications

In this section, we illustrate potential applications of collage theorems through concrete examples. While fixed point theorems provide both theoretical and practical tools for establishing the existence of solutions to a given system, model, or optimization problem, collage theorems enable us to reverse this perspective. In a broader sense, collage theorems constitute a powerful framework for representing complex systems by simpler ones, analyzing the behavior of iterative processes, and predicting suitable systems or models via reverse engineering from an optimal or admissible solution.

This methodology is based on extrapolating from a prescribed solution, together with certain parameters, in order to identify an appropriate underlying model. Its scope is considerable: beyond enhancing our understanding of complex phenomena, it offers practical advantages for the design, approximation, and optimization of systems across a wide range of applications.

To make this approach more explicit, we begin with a simple setting. Consider

$$X = \left\{ x \in C[a,b] \,:\, \|x\|_\infty = \sup_{t\in[a,b]} |x(t)| < K \right\} \subseteq C[a,b],$$

endowed with the metric

$$d_2(x,y) = \left(\int_a^b \big(x(s) - y(s)\big)^2 \,\mathrm{d}s \right)^{1/2}.$$

The metric space (X, d_2) is not complete, since the completion of $(C[a,b], d_2)$ is $L^2[a,b]$. However, completeness is not essential in the present context. For the purpose of constructing a model system corresponding to a prescribed target solution, it suffices to work within the space (X, d_2).

Example 5.1. We begin by considering the nonlinear mapping

$$T(x) = \Big(\frac{1}{2}x\,\mathrm{e}^{-x} + 1\Big)\cos x,$$

defined on the interval $[0, 3/2]$. It is readily observed that the operator $T : [0,3/2] \to [0,3/2]$ is not a contraction in the classical sense on this interval, as illustrated in Figure 1 (left). Nevertheless, when analyzed within the framework of enriched contractions, the mapping T exhibits a $(0.5, 1)$-enriched contraction behavior on $[0, 3/2]$, as shown in Figure 1 (right).

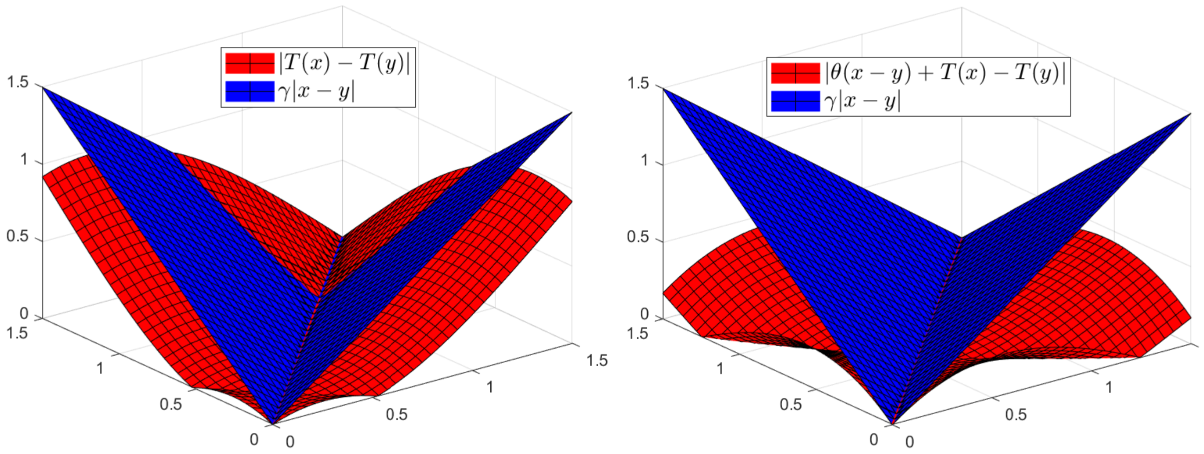

Figure 1: (left) Graph of the mapping $T(x) = \left(\frac{1}{2}x\,\mathrm{e}^{-x}+1\right)\cos x$ on $[0, 3/2]$, demonstrating that T does not satisfy the classical contraction condition; (right) Graphical illustration of the (γ, θ)-enriched contraction property of T on $[0, 3/2]$ with parameters $\gamma = 0.5$ and $\theta = 1$, highlighting its enriched-contractive behavior.

Let $X = \{x_1, x_2, \dots, x_m\}$ be a finite set of real numbers randomly sampled from the interval $[0, 3/2]$, where m is fixed (in our numerical experiments, $m = 100$). We denote by $Y = T(X)$ the image of X under T. Accordingly, for each realization we obtain data sets of the form

$$X_j = \left\{(x, y) : x \in X,\ y \in Y\right\}, \qquad j = 1, 2, 3.$$

Our objective is to approximate the operator T by a polynomial mapping that fits the given data and exhibits an approximately contractive behavior.

To this end, we consider a polynomial of degree five,

$$P(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,$$

and define the associated squared ℓ_2 fitting error by

$$d_2(T(X), P(X))^2 = \sum_{x\in X}\left(T(x) - P(x)\right)^2.$$

Equivalently,

$$d_2(T(X), P(X))^2 = \sum_{x\in X}\left(\left(\tfrac{1}{2}x\,e^{-x} + 1\right)\cos x - \left(a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0\right)\right)^2. \tag{16}$$

By minimizing the functional (16) with respect to the coefficients a_i, $0 \le i \le 5$, using MATLAB, we obtain the values reported in Table 2 for each data set X_j, $j = 1, 2, 3$.

Table 2: Values of the coefficients a_i obtained by minimizing the fitting function (16) for different data sets X_j, $j = 1, 2, 3$, as well as the associated squared ℓ_2 fitting error and the maximal absolute error on $[0, 3/2]$

	a_0	a_1	a_2	a_3	a_4	a_5	$d_2(\mathbb{T}(X_j), P(X_j))$	max err
X_1	0.999901	0.503012	−1.02180	0.0620337	0.124521	−0.0280100	2.8343×10^{-4}	3.130×10^{-4}
X_2	0.999949	0.502383	−1.01850	0.0555207	0.129833	−0.0295300	2.8965×10^{-4}	2.987×10^{-4}
X_3	0.999880	0.503226	−1.02212	0.0621604	0.124518	−0.0280039	3.2109×10^{-4}	3.410×10^{-4}

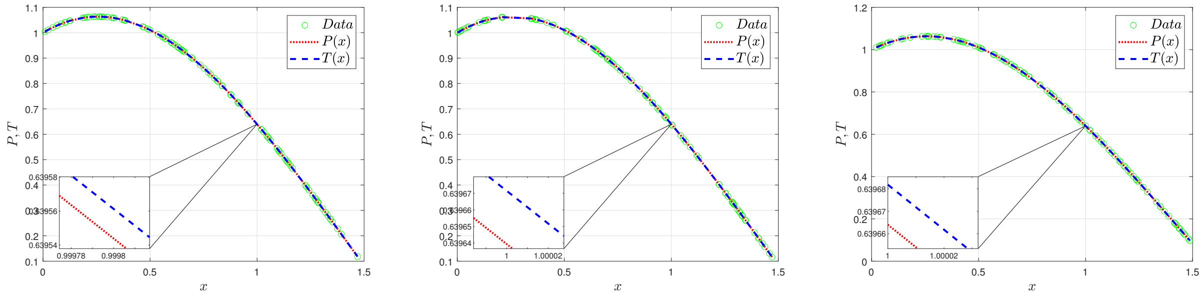

Figure 2: Comparison of the operator T and the fitted polynomial $P(x)$ for each data set X_j, $j = 1, 2, 3$

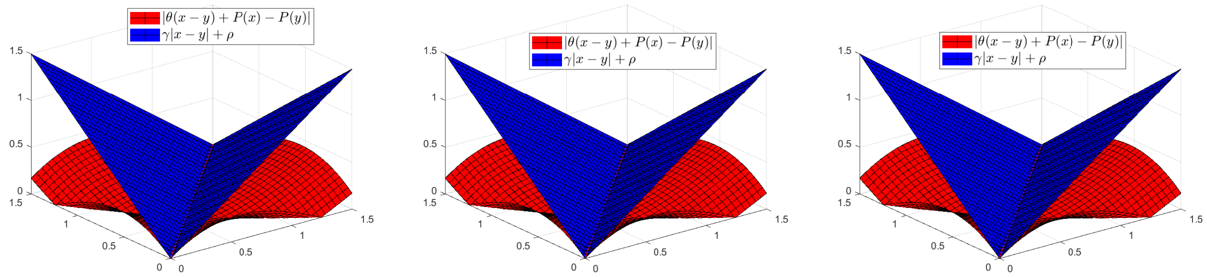

Figure 3: Graphical illustration of the polynomial mapping P, showing that for every $x \in X$, $P(x)$ satisfies the conditions of a ρ-roughly $(0.2, 1.2)$-enriched contraction on $[0, 3/2]$

For each data set X_j, the corresponding polynomial $P(x)$ provides a close approximation to the operator $T(x)$, as illustrated in Figure 2. In all cases, the fitted polynomials closely track the behavior of T over the interval $[0, 3/2]$.

The polynomials provide accurate approximations of T across the entire interval.

Finally, Figure 3 demonstrates that, for every $x \in X$, the mapping $P(x)$ satisfies the conditions of a ρ-roughly $(0.2, 1.2)$-enriched contraction. Here, the roughness parameter is defined by

$$\rho = d_2(X, P(X)) = \left(\sum_{x \in X} \left(x - \left(a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \right) \right)^2 \right)^{1/2},$$

which measures the deviation between the identity mapping and the polynomial operator P with respect to the ℓ_2-metric.

Example 5.2. *Let*

$$y_p(t) = 2t - t^2 + 1$$

be a prescribed target function, which is the exact solution of the boundary value problem

$$4y''(t) + (y'(t))^2 + 4y(t) = 0, \qquad y(0) = 1,\ y(1) = 2. \tag{17}$$

In order to construct an associated fixed-point formulation, we first rewrite (17) *in the form*

$$y''(t) = -\frac{1}{4}\Big((y'(t))^2 + 4y(t) \Big).$$

Integrating twice, with using boundary conditions we arrived to

$$y(t) = 1 + t + \frac{1}{4}\int_0^1 G(t,s)\left((y'(s))^2 + 4y(s)\right)\mathrm{d}s, \tag{18}$$

where

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}$$

Equivalently, it can be obtained using the Green function $G(t,s)$, associated with the linear problem $u'' = f(t)$ (or $u'' = f(t)$), with boundary conditions $u(0) = u(1) = 0$.

By (18) *we define the operator $T : X \to X$ as*

$$(Ty)(t) = 1 + t + \frac{1}{4}\int_0^1 G(t,s)\left((y'(s))^2 + 4y(s)\right)\mathrm{d}s,$$

where

$$X = \left\{y \in C^1[0,1] : y(0) = 1,\ y(1) = 2\right\}.$$

Then $y \in X$ is a fixed point of T if and only if it is a classical solution of (17). *In particular, the function $y_p(t) = 2t - t^2 + 1$ is a fixed point of T.*

Consider the nonlinear boundary value problem

$$y''(t) + b(y'(t))^2 + cy(t) = 0, \qquad y(0) = 1,\ y(1) = 2, \tag{19}$$

which is structurally similar to (17). *Its solution is the fixed point of the operator*

$$(T_2y)(t) = 1 + t + \int_0^1 G(t,s)\left(b(y'(s))^2 + cy(s)\right)\mathrm{d}s,$$

defined on

$$X = \left\{y \in C^1[0,1] : y(0) = 1,\ y(1) = 2,\ \|y\|_\infty \le M\right\},$$

for a suitable constant $M > 0$.

For any $y, z \in X$ and $t \in [0,1]$, we have

$$\left|(T_2y)(t) - (T_2z)(t)\right| \le \int_0^1 |G(t,s)|\Big(|b|\left|(y'(s))^2 - (z'(s))^2\right| + |c|\,|y(s) - z(s)|\Big)\mathrm{d}s.$$

Using the estimate

$$\left|(y')^2 - (z')^2\right| \le 2M|y' - z'|,$$

together with

$$\sup_{t\in[0,1]}\int_0^1 |G(t,s)|\,\mathrm{d}s = \frac{1}{8},$$

we obtain

$$\|T_2y - T_2z\|_\infty \le \frac{1}{8}\Big(2M|b| + |c|\Big)\|y - z\|_\infty.$$

Thus, T_2 *is a* $(0,\, 2M|b| + |c|)$*-enriched contraction whenever* $2M|b| + |c| < 8$.

Since $C[0,1] \subset L^2[0,1]$, *the Collage distance is given by*

$$d_2(y_p, T_2y_p)^2 = \int_0^1 \Big(y_p(t) - (T_2y_p)(t)\Big)^2 \,\mathrm{d}t$$
$$= \frac{b^2}{81} + \frac{19bc}{567} - \frac{5b}{126} + \frac{1097c^2}{45360} - \frac{143c}{2520} + \frac{1}{30}.$$

Minimization with respect to b and c yields

$$b = \frac{1}{4}, \qquad c = 1.$$

Consequently, (19) *reduces to*

$$y''(t) + \frac{1}{4}(y'(t))^2 + y(t) = 0, \qquad y(0) = 1,\ y(1) = 2,$$

whose exact solution is

$$\overline{y}_p(t) = 2t - t^2 + 1.$$

In this case,

$$d_2(y_p, T_2y_p) = 0, \qquad d_2(y_p, \overline{y}_p) = 0.$$

Example 5.3. *Let*

$$y_p(t) = t^2$$

be a target function and consider the nonlinear boundary value problem

$$y''(t) + b\,(y'(t))^4 + c\,y(t)^2 + d = 0, \qquad y(0) = 0,\ y(1) = 1. \tag{20}$$

Its solution is the fixed point of the operator $T : X \to X$ *defined by*

$$(Ty)(t) = t - \int_0^1 G(t,s)\Big(b\,(y'(s))^4 + c\,y(s)^2 + d\Big)\,\mathrm{d}s,$$

where

$$X = \Big\{y \in C[0,1] : y(0) = 0,\ y(1) = 1,\ \|y\|_\infty \le N,\ \|y'\|_\infty \le M\Big\}.$$

and

$$G(t,s) = \begin{cases} t(1-s), & t \le s, \\ s(1-t), & s \le t. \end{cases}$$

One obtains the estimate

$$\|Ty - Tz\|_\infty \le \frac{1}{8}\Big(8|b|M^3 + 2|c|N\Big)\|y - z\|_\infty,$$

and hence T is a

$\Big(0,\ |b|M^3 + \frac{|c|N}{4}\Big)$*-enriched contraction* *whenever*

$$|b|M^3 + \frac{|c|N}{4} < 1.$$

Since $C[0,1] \subset L^2[0,1]$, *the Collage distance is given by*

$$d_2(y_p, T_2 y_p)^2 = \frac{16b^2}{351} + \frac{2bc}{351} + \frac{bd}{27} + \frac{2b}{27} + \frac{c^2}{5616} + \frac{cd}{432} + \frac{c}{216} + \frac{d^2}{120} + \frac{d}{30} + \frac{1}{30}$$

Minimization with respect to b and c yields

$$b = 1, \qquad c = -16, \qquad d = -2,$$

Consequently, (19) *reduces to*

$$y''(t) + (y'(t))^4 - 16\, y(t)^2 - 2 = 0, \qquad y(0) = 1,\ y(1) = 2,$$

whose exact solution is

$$\overline{y}_p(t) = t^2$$

In this case,

$$d_2(y_p, T_2 y_p) = 0, \qquad d_2(y_p, \overline{y}_p) = 0.$$

6. Conclusion

This work has undertaken a comprehensive exploration of enriched contraction mappings, extending the classical Banach contraction principle to a broader and more versatile framework. Through a systematic investigation, we have established several foundational results that not only deepen the theoretical understanding of these mappings but also significantly expand their applicability to inverse problems and practical scenarios.

Theorems on well-posedness and Ulam-Hyers stability for the fixed point problem associated with (θ, γ)-enriched contractions ensure that solutions are not only unique and existent but also robust under perturbations. This stability is crucial for applications where data or models are subject to noise and uncertainty. Furthermore, the derived collage-type theorems provide a powerful tool for inverse problems, allowing the reformulation of complex fixed-point equations into tractable optimization problems. By minimizing the collage distance, one can efficiently approximate solutions without directly solving the often intractable fixed-point equation, thus offering a practical methodology for applications ranging from fractal image coding to parameter estimation in dynamical systems.

The introduction of roughly enriched contractions, which accommodate additive perturbations, further generalizes the framework, enabling the analysis of mappings that lack strict contractivity. This extension is particularly valuable in real-world applications where perfect contractivity is an idealized assumption. The Krasnosel'ski iteration scheme, analyzed in this context, provides a iterative method for finding approximate fixed points, with explicit error bounds and convergence guarantees, even for discontinuous or highly irregular mappings.

The practical efficacy of the theoretical developments has been demonstrated through diverse applications, including the modeling of non-linear systems and the solution of inverse problems. These examples underscore the potential of enriched contractions to address challenges in various fields, such as engineering, physics, and data science, where traditional contraction mappings may be too restrictive.

In conclusion, this work bridges theoretical advances with practical applications, offering a robust and flexible framework for solving fixed-point problems. The generalizations presented here open new avenues for research and application, promising to enhance the toolkit available for tackling complex problems in mathematical analysis and beyond. Future directions may include further extensions to multi-valued mappings, applications in machine learning and neural networks, and explorations in infinite-dimensional spaces. The continued development of this paradigm holds the potential to yield even more sophisticated and efficient methods for understanding and solving nonlinear problems.

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